



On the Equations Defining Abelian Varieties. II

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On the Equations Defining Abelian Varieties. II

D. MUMFORD (Cambridge, Mass.)

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In the first part of this paper, we have analyzed a single abelian variety X . In particular, if L is an ample invertible sheaf on X , we have analyzed the vector space $\Gamma(X, L)$ and the ring

$$\bigoplus_n \Gamma(X, L^n)$$

and have shown 1) how to choose canonical bases of these vector spaces, 2) how to express this ring as a quotient of a polynomial ring by an explicit homogeneous ideal involving coefficients which are essentially the “theta-null werte” of X . In this second part, we shall apply the first part to embed both the moduli spaces of abelian varieties, and the inverse limit of these spaces over successively higher levels, as open sets in projective schemes associated to homogeneous coordinate rings defined by *explicit* homogeneous ideals. We also introduce algebraic theta functions, defined on a 2-adic vector space in terms of which our results on moduli take on a simple form.

I want to offer some explanation of why the 2-adics play such a central role in this theory. The situation is this: if you stick to abelian varieties of char. p ($p \neq 2$), then you can build up a theory of theta functions for these over any (restricted) product

$$\prod'_{l \in S} \mathbb{Q}_l$$

where S is any set of primes containing 2, but not containing p . In other words, \mathbb{Q}_2 always has to be there, but you can throw in plenty of other factors if you like. Using only \mathbb{Q}_2 seemed to have two advantages: (i) you can deal simultaneously with all characteristics except 2, and (ii) the resulting theta-functions are more concise, i.e., are defined on the smallest locally compact group which admits them. I might have written a *general* theory for some arbitrary set S — clearly this is the accepted French approach — but there seemed no point in not sticking to the simplest and most basic case. The essential features are related to the fact that multiplication by 2 does not preserve Haar measure.

§ 6. Structure of the Moduli Space

To study questions of moduli, we must first have a theory of families of the objects to be classified. Therefore, we must generalize our theory to abelian schemes:

Definition. Let S be a scheme. An *abelian scheme* \mathcal{X} over S is a group scheme \mathcal{X} finitely presented over S such that the projection $\pi: \mathcal{X} \rightarrow S$ is proper and smooth and the geometric fibres of π are connected.

For some of the basic facts about abelian schemes, we refer the reader to [9], Ch. 6¹.

Definition. Let L be an invertible sheaf on S . Then 1. $H(L)$ is the group of sections $\alpha: S \rightarrow \mathcal{X}$ of π such that if $T_\alpha: \mathcal{X} \rightarrow \mathcal{X}$ is translation by α , then

$$T_\alpha^* L \cong L \otimes \pi^* M$$

for some invertible sheaf M on S .

2. $H_0(L)$ is the subgroup of those α such that

$$T_\alpha^* L \cong L.$$

3. $\mathcal{G}(L)$ is the group of pairs (α, φ) where $\alpha \in H_0(L)$ and $\varphi: L \rightarrow T_\alpha^* L$ is an isomorphism.

Following a familiar procedure, we note first that H and \mathcal{G} can be extended to *functors* from the category of S -schemes to the category of groups:

Definition. For all S -schemes $f: T \rightarrow S$, let $H^f(L) = H(L')$, $\mathcal{G}^f(L) = \mathcal{G}(L')$, where if F is the morphism in the diagram:

$$\begin{array}{ccc} \mathcal{X} \times_S T & \xrightarrow{F} & \mathcal{X} \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S \end{array}$$

then $L' = F^* L$. Then $f \mapsto H^f(L)$, $f \mapsto \mathcal{G}^f(L)$ are functors in an obvious way.

Proposition 1. Assume that L is relatively ample over S . Then the 2 functors $f \mapsto H^f$ and $f \mapsto \mathcal{G}^f$ are representable by group schemes $\underline{H}(L)$, $\underline{\mathcal{G}}(L)$ flat and of finite presentation over S . $\underline{H}(L)$ is a closed sub-group scheme of \mathcal{X} itself, finite over S , and there is a canonical exact sequence:

$$0 \rightarrow G_{m,S} \rightarrow \underline{\mathcal{G}}(L) \rightarrow \underline{H}(L) \rightarrow 0.$$

¹ If any doubt should arise as to whether results in [9] are still valid if S is non-noetherian, it should be dispelled by noticing that any abelian scheme over an affine S is obtained by base extension from an abelian scheme over $\text{Spec}(R)$, where R is a \mathbb{Z} -algebra of finite type, [5], Ch. 4, §§ 8, 11.

Proof. First of all, $\underline{H}(L)$ is nothing but the kernel of the canonical homomorphism: $\Lambda(L): \mathcal{X} \rightarrow \hat{\mathcal{X}}$. $\Lambda(L)$ is flat and finite and of finite presentation (cf. [9], p. 122), hence $\underline{H}(L)$ is flat and finite and finitely presented over S . Secondly, the morphism from the functor \mathcal{G}^f to the functor H^f is represented (relatively) by G_m -bundles. In fact, if $\alpha: T \rightarrow \mathcal{X} \times_S T$ is an element of $H^f(T)$, and if

$$M = \pi_*(T_\alpha^*(L) \otimes L'^{-1})$$

[here L' is the sheaf on $\mathcal{X} \times_S T$ induced by L and π is the projection from $\mathcal{X} \times_S T$ to T], then

$$T_\alpha^* L' \cong L' \otimes \pi^* M,$$

and multiplication by non-zero sections of M defines the isomorphisms from L' to $T_\alpha^* L'$. Therefore the relative functor in this case is represented by the line bundle M on T corresponding to M^{-1} , minus its 0-section. *Q.E.D.*

Definition. Let $\delta = (d_1, d_2, \dots, d_g)$ be any set of elementary divisors (d_i integers, ≥ 1 , $d_{i+1} | d_i$). Define a functor $\mathcal{G}(\delta)$ on the category of all schemes S to the category of groups by:

$\mathcal{G}^S(\delta) =$ group of triples (α, x, l) , $\alpha \in \Gamma(S, \mathcal{O}_S^*)$,

x is a map from the set $\pi_0(X)$ of connected components of X to $K(\delta)$, the discrete group $\oplus \mathbb{Z}/d_i \mathbb{Z}$.

$l = (l_1, \dots, l_g)$, where l_i is a d_i -th root of 1 in $\Gamma(S, \mathcal{O}_S^*)$.

Multiplication is:

$$(\alpha, x, l) \cdot (\alpha', x', l') = (\alpha \cdot \alpha' \cdot l'(x), x + x', l + l')$$

where

$$l + l' = (l_1 \cdot l'_1, \dots, l_g \cdot l'_g)$$

$$l'(x), \text{ on the component } Y, = \prod_{i=1}^g l_i^{a_i}$$

$$\text{if } x(Y) = (a_1, \dots, a_g).$$

[We add the l 's instead of multiplying them to be consistent with our previous notation.]

It is easy to check that $S \mapsto \mathcal{G}^S(\delta)$ is represented by a group scheme $\underline{\mathcal{G}}(\delta)$, flat and of finite type over \mathbb{Z} , that fits into a canonical exact sequence:

$$0 \rightarrow G_m \rightarrow \underline{\mathcal{G}}(\delta) \rightarrow \underline{H}(\delta) \rightarrow 0$$

of group schemes, where

$$\underline{H}(\delta) = [\oplus_i \mathbb{Z}/d_i \mathbb{Z}] \oplus [\oplus_i \mu_{d_i}]$$

and where $\mathbb{Z}/d_i \mathbb{Z}$ is taken as a discrete reduced group scheme and μ_{d_i} is the usual group scheme of d_i -th roots of 1.

Definition. A \mathcal{G} -structure on a relatively ample invertible sheaf L is an isomorphism over S of the group schemes $\mathcal{G}(L)$ and $\mathcal{G}(\delta) \times S$, for some δ , which is the identity on the sub-group schemes $G_{m,S}$. When this exists, δ is called the *type* of L .

In this definition, we have included some types of non-separable invertible sheaves [in fact, all ample invertible sheaves on an abelian variety with p^g points of order p have a type in the above sense] because in the present categorical approach it is no trouble. However, this was just for the fun of it and we shall now restrict ourselves to the separable case.

Fix $\delta = (d_1, \dots, d_g)$. Assume all d_i even. Assume all schemes are schemes over $\text{Spec } \mathbb{Z}[d^{-1}]$, where

$$d = \prod_{i=1}^g d_i.$$

Definition. An invertible sheaf L on \mathcal{X}/S is *symmetric* if $\iota^* L \cong L$, where $\iota: \mathcal{X} \rightarrow \mathcal{X}$ is the inverse. It is *totally symmetric* if there is an isomorphism $\varphi: L \xrightarrow{\sim} \iota^* L$ which restricts to the identity on $L \otimes \underline{O}_{\mathcal{X}_2}$, where $\mathcal{X}_2 \subset \mathcal{X}$ is the kernel of 2δ . It is *normalized* if $\varepsilon^* L \cong \underline{O}_S$ where $\varepsilon: S \rightarrow \mathcal{X}$ is the identity section.

Definition. Let $\lambda: \mathcal{G}(L) \xrightarrow{\sim} \mathcal{G}(\delta) \times S$ be a \mathcal{G} -structure for a symmetric relatively ample invertible sheaf L on \mathcal{X} . Let $\psi: L \xrightarrow{\sim} \iota^* L$ be any isomorphism. We define:

i) an automorphism δ_{-1} of the functor $f \rightarrow \mathcal{G}^f(L)$. Given $f: T \rightarrow S$, let $L' = L \otimes \underline{O}_T$ be the induced sheaf on $\mathcal{X} \times_S T$. Let $\alpha: T \rightarrow \mathcal{X} \times_S T$ be a section and

$$\varphi: L' \xrightarrow{\sim} T_{\alpha}^*(L')$$

an isomorphism, so that $(\alpha, \varphi) \in \mathcal{G}^f(L)$. Let $\psi': L' \rightarrow \iota^* L'$ be the isomorphism induced by ψ . Then

$$\delta_{-1}((\alpha, \varphi)) = (\iota \circ \alpha, (T_{\iota \circ \alpha}^* \psi')^{-1} \circ \iota^* \varphi \circ \psi'),$$

i.e.,

$$\begin{array}{c} L' \xrightarrow{\psi'} \iota^* L' \xrightarrow{\sim} \iota^* (T_{\alpha}^* L') \\ \parallel \\ T_{\iota \circ \alpha}^* (\iota^* L') \xleftarrow{\sim} T_{\iota \circ \alpha}^* L', \end{array}$$

where $\iota: \mathcal{X} \times_S T \rightarrow \mathcal{X} \times_S T$ is the inverse.

ii) The automorphism δ_{-1} of the functor induces an automorphism $\underline{\delta}: \mathcal{G}(L) \rightarrow \mathcal{G}(L)$ of the scheme.

iii) Similarly, the map

$$(\alpha, x, l) \mapsto (\alpha, -x, -l)$$

where if $l=(l_1, \dots, l_g)$, then $-l=(l_1^{-1}, \dots, l_g^{-1})$ gives an automorphism D_{-1} of the functor $S \rightarrow \mathcal{G}^S(\delta)$.

iv) This induces an automorphism D of the group scheme $\underline{\mathcal{G}}(\delta)$.

v) Then λ is *symmetric* if $\underline{D} \circ \lambda = \lambda \circ \underline{\delta}$.

Definition. We consider triples of the following type:

- i) an abelian scheme \mathcal{X} over S ,
- ii) a relatively ample, totally symmetric, normalized invertible sheaf L on \mathcal{X} ,
- iii) a symmetric \mathfrak{g} -structure $\lambda: \underline{\mathcal{G}}(L) \xrightarrow{\sim} \underline{\mathcal{G}}(\delta) \times S$ for L .

We shall call this triple an *abelian scheme with a δ -marking*.

Definition. For all schemes S , let $\mathcal{M}_\delta(S)$ denote the set of abelian schemes \mathcal{X} over S with δ -markings, taken modulo isomorphisms. As S varies, these sets form a functor \mathcal{M}_δ in S . This will be called the *moduli functor for abelian schemes with δ -marking*.

The object of this section will be to show that \mathcal{M}_δ is representable, and to represent it by an open subset of a definite projective variety. The next step is to study the representations of $\mathcal{G}(\delta)$.

Definition. Let V_δ be the free $\mathbf{Z}[d^{-1}]$ -module of functions from $K(\delta)$ to $\mathbf{Z}[d^{-1}]$. Then for all schemes S , and invertible sheaves L on S , $V_\delta \otimes_{\mathbf{Z}} L$ is the sheaf of functions from $K(\delta)$ to L . The discrete group $\mathcal{G}^S(\delta)$ acts \underline{O}_S -linearly on this sheaf, exactly as $\mathcal{G}(\delta)$ acted on V_δ in § 1: e.g., if $(\alpha, x, l) \in \mathcal{G}^S(\delta)$,

$$f \in \Gamma(U, V_\delta \otimes_{\mathbf{Z}} L).$$

Assume for simplicity that U is connected, that $x(U) = (a_1, \dots, a_g)$ and that $l = (l_1, \dots, l_g)$. Then (α, x, l) takes f into f^* , where

$$f^*(b_1, \dots, b_g) = \alpha \cdot \prod l_i^{b_i} \cdot f(a_1 + b_1, \dots, a_g + b_g).$$

Let $E_\delta = V(V_\delta)$: this is a vector bundle of rank d over $\text{Spec } \mathbf{Z}[d^{-1}]$, and it is a direct sum of trivial line bundles L_a with canonical sections $[a]$: Recall that, in general, an S -valued point of E_δ corresponds to a homomorphism from V_δ to $\Gamma(S, \underline{O}_S)$. Then $[a]$ corresponds to the homomorphism

$$f \mapsto f(a)$$

from $V(\delta)$ to $\mathbf{Z}[d^{-1}]$. All the actions of $\mathcal{G}^S(\delta)$ on $V_\delta \otimes \underline{O}_S$ can be put together dually in an anti-representation of $\underline{\mathcal{G}}(\delta)$ on E_δ , i.e., a representation in which the order of multiplication is reversed. This is clear from

a functorial point of view, but it may be useful to define this anti-representation directly by putting together representations on the various subgroups:

the subgroup G_m , with S -valued points $(\alpha, 0, 0)$, acts by homotheties on E_δ ,

the discrete subgroup $\oplus \mathbb{Z}/d_i \mathbb{Z}$, with S -valued points $(1, a, 0)$, acts by permuting the sections $[a]$: thus the point b takes $[a]$ to $[a+b]$,

the discrete but twisted subgroup $\oplus \mu_{d_i}$ with S -valued points $(1, 0, l)$, acts diagonally: thus the point (l_1, \dots, l_g) takes $[a]$ to $\prod l_i^{a_i} \cdot [a]$.

Proposition 2. *Let S be a scheme and let*

$$\rho: [\mathcal{G}(\delta) \times S] \times_S F \rightarrow F$$

be an anti-representation over S of $\mathcal{G}(\delta) \times S$ on a vector bundle F over S of rank d . Assume that the subgroup G_m acts on F in the standard way. Then there is a line bundle L over S and an isomorphism

$$F \xrightarrow{\sim} [E_\delta \times S] \otimes L$$

(where \otimes denotes tensor product of vector bundles over S) such that the action of $\mathcal{G}(\delta)$ on F corresponds to the above action of $\mathcal{G}(\delta)$ on E_δ tensored with the trivial action on L . Moreover, this isomorphism is unique up to multiplication by an element of $\Gamma(S, \mathcal{O}_S^)$.*

Proof. This is nearly the same as that of Prop. 3, § 1, except that we must use a basic result on the representations of μ_d established in [4]. It is shown there that if a group scheme of the form $\oplus \mu_{d_i}$ is represented, over S , in a vector bundle F , then F is a direct sum of sub-vector bundles F_a , $a \in K(\delta)$, where the group acts on F_a by the character

$$\begin{aligned} \oplus \mu_{d_i} &\rightarrow G_m \\ (l_1, \dots, l_g) &\mapsto \prod_{i=1}^g l_i^{a_i}. \end{aligned}$$

Now we realize $\oplus \mu_{d_i}$ as the subgroup of $\mathcal{G}(\delta)$ of triples $(1, 0, l)$; and decompose F accordingly. Exactly as in §1, the action of the $\mathbb{Z}[d^{-1}]$ -valued points $\sigma_a = (1, a, 0)$ of $\mathcal{G}(\delta)$ permutes these F_a transitively. Therefore all F_a are non-empty; since the rank of F is d , each F_a is a line bundle over S . Let $L = F_0$. We set up the required isomorphism by first identifying

$$F_0 = L \cong (L_0 \times S) \otimes L$$

then identifying F_a with $(L_a \times S) \otimes L$ by using the actions of σ_a and the first identification; and then taking the direct sum. The details work out exactly as in §1. *Q.E.D.*

Now start with an abelian scheme \mathcal{X}/S with δ -marking. Let L be the given sheaf on \mathcal{X} , and let $F = V(\pi_* L)$: this is a vector bundle over S of rank d . Let $f: T \rightarrow S$ be a morphism of schemes. Then a T -valued point of F/S , i.e., a morphism g is the diagram:

$$\begin{array}{ccc} T & \xrightarrow{g} & F \\ & \searrow f & \swarrow \text{canonical morphism} \\ & S & \end{array}$$

is the same thing as a homomorphism γ :

$$f^*(\pi_* L) \xrightarrow{\gamma} \underline{O}_T$$

of \underline{O}_T -modules. I claim that $\mathcal{G}(L)$ is anti-represented over S on the vector bundle F in a canonical way. In fact, if $f: T \rightarrow S$ is a morphism of schemes, a T -valued point of $\mathcal{G}(L)/S$ is given by a section

$$\alpha: T \rightarrow \mathcal{X} \times_S T$$

and an isomorphism

$$\varphi: L' \xrightarrow{\sim} T_\alpha^* L'$$

if L' is the induced sheaf on $\mathcal{X} \times_S T$. Then φ induces

$$\begin{array}{ccc} \pi_* L' & \xrightarrow{\varphi^*} & \pi_*(T_\alpha^* L') \\ \wr \parallel & & \wr \parallel \\ f^*(\pi_* L) & & (\pi \circ T_\alpha)_*(T_\alpha^* L') \\ & & \wr \parallel \\ & & \pi_*(T_\alpha)_* T_\alpha^*(L') \\ & & \wr \parallel \\ & & \pi_* L' \\ & & \wr \parallel \\ & & f^*(\pi_* L) \end{array}$$

by standard canonical identifications. Let the composite isomorphism on $f^*(\pi_* L)$ be called $[\varphi]$. Then this acts on a T -valued point $\gamma: f^*(\pi_* L) \rightarrow \underline{O}_T$ of F/S by taking it to the new point $\gamma \circ [\varphi]$. This gives us an anti-representation of the functor $\mathcal{G}(L)$ on the functor associated to F : hence an anti-representation of the scheme $\mathcal{G}(L)$ on F .

But \mathcal{X}/S has a given δ -marking. Hence we also have an anti-representation over S of $\mathcal{G}(\delta) \times S$ on F . Applying Proposition 2, we get an isomorphism

$$F \xrightarrow{\sim} (E_\delta \times S) \otimes L,$$

for some line bundle L on S , unique up to multiplication by elements of $\Gamma(S, \mathcal{O}_S)$. In terms of sheaves, this gives us an isomorphism

$$\pi_* L \xleftarrow{\sim} V_\delta \otimes_{\mathbf{Z}} K$$

for some invertible sheaf K on S , unique up to multiplication by elements of $\Gamma(S, \mathcal{O}_S)$. The fact that the isomorphism of F and $(E_\delta \times S) \otimes L$ commutes with pair of actions of $\mathcal{G}(L)$ and $\mathcal{G}(\delta)$ gives a corresponding fact on the sheaf level:

for all $f: T \rightarrow S$, let the \mathcal{G} -structure induce an isomorphism

$$\mathcal{G}(L) = \mathcal{G}^f(L) \cong \mathcal{G}^T(\delta)$$

where L' is the induced sheaf on $\mathcal{X} \times_S T$. Then the action of $\mathcal{G}(L')$ on $\pi_* L'$ — which is canonically isomorphic to $f^* \pi_* L$ — goes over under this isomorphism to the action of $\mathcal{G}^T(\delta)$ on $V_\delta \otimes_{\mathbb{Z}} f^* K$.

It follows, incidentally, that the above isomorphism of $\pi_* L$ and $V_\delta \otimes_{\mathbb{Z}} K$ determines the isomorphism of $\mathcal{G}(L')$ and $\mathcal{G}^T(\delta)$ for all f : hence it determines the given isomorphism of $\mathcal{G}(L)$ and $\mathcal{G}(\delta) \times S$. Now assume L is relatively very ample: this occurs if all d_i are divisible by 4 for example. Then we get a closed immersion over S :

$$i: \mathcal{X} \hookrightarrow P(\pi_* L) \cong P[V_\delta \otimes_{\mathbb{Z}} K] \cong P(V_\delta) \times S,$$

which is determined by the original δ -marking on \mathcal{X} . Conversely, this immersion determines the δ -marking: the sheaf L , for example, is obtained

a) by pulling back $\mathcal{O}_P(1)$ via $p_1 \circ i: \mathcal{X} \rightarrow P(V_\delta)$,

b) by then normalizing this sheaf on the identity section. In other words

$$(*) \quad L \cong (p_1 \circ i)^* [\mathcal{O}_P(1)] \otimes (p_1 \circ i \circ \varepsilon \circ \pi)^* [\mathcal{O}_P(-1)].$$

And the morphism from $\mathcal{G}(\delta) \times S$ to $\mathcal{G}(L)$ is determined as follows: let $K = (p_1 \circ i \circ \varepsilon)^* (\mathcal{O}_P(-1))$. Then the isomorphism (*) determines the isomorphism:

$$\begin{aligned} \pi_* L &\cong \pi_* [(p_1 \circ i)^* (\mathcal{O}_P(1))] \otimes K \\ &\cong p_{2,*} [\mathcal{O}_P(1) \otimes \mathcal{O}_S] \otimes K \\ &\cong V_\delta \otimes_{\mathbb{Z}} K, \end{aligned}$$

which, as we just saw, determines the \mathcal{G} -structure. Summarizing, the abelian scheme \mathcal{X}/S with δ -marking determines, and is determined by the closed immersion:

$$i: \mathcal{X} \hookrightarrow P(V_\delta) \times S.$$

Recall that the group structure on \mathcal{X} is determined completely by the identity section $\varepsilon: S \rightarrow \mathcal{X}$, ([9], p. 117). Therefore, the whole functor \mathcal{M}_δ is isomorphic to a subfunctor of the functor $\text{Hilb}_P^{(1)}(V_\delta)$ of all flat fa-

milies of closed subschemes of $P(V_\delta)$ with distinguished section (cf. [9], Ch. 0, § 5).

In particular, we have defined, from \mathcal{X}/S with δ -marking, the S -valued point of $P(V_\delta)$: $p_1 \circ i \circ \varepsilon: S \rightarrow P(V_\delta)$. It is not hard to verify that this defines a morphism of functors:

$$\mathcal{M}_\delta \xrightarrow{t} h_{P(V_\delta)}.$$

The next step is to describe the image. First define a closed subscheme $\bar{M}_\delta \subset P(V_\delta)$ by RIEMANN'S theta relations:

let $Q(a) \in V_\delta$ be the function which is 1 at a , and 0 elsewhere;

let $Z_2 \subset K(\delta)$ be the subgroup of points of order 2; embed $K(\delta) \subset K(2\delta)$ as before.

1. For all elements $a, b, c, d \in K(2\delta)$ such that they are all congruent modulo $K(\delta)$, and for all $l \in Z_2$, set

$$\begin{aligned} & \left[\sum_{\eta \in Z_2} l(\eta) \cdot Q(a+b+\eta) \cdot Q(a-b+\eta) \right] \times \\ & \quad \times \left[\sum_{\eta \in Z_2} l(\eta) \cdot Q(c+d+\eta) \cdot Q(c-d+\eta) \right] - \\ & \quad - \left[\sum_{\eta \in Z_2} l(\eta) \cdot Q(a+d+\eta) \cdot Q(a-d+\eta) \right] \times \\ & \quad \times \left[\sum_{\eta \in Z_2} l(\eta) \cdot Q(c+b+\eta) \cdot Q(c-b+\eta) \right] = 0; \end{aligned}$$

2. For all $a \in K(\delta)$, set

$$Q(a) - Q(-a) = 0.$$

The main result can now be stated:

Theorem. *If d_1, \dots, d_g are all divisible by 8, then there is an open subset M_δ of \bar{M}_δ such that t defines an isomorphism of \mathcal{M}_δ with the functor of points of the subscheme M_δ of $P(V_\delta)$.*

We can go even further: not only can we identify the moduli scheme M_δ , but we can write down the universal projective family of abelian varieties over M_δ obtained by the immersion i . Define

$$\bar{A}_\delta \subset P(V_\delta) \times \bar{M}_\delta$$

by the relations:

let X_a be the same function as $Q(a)$, but in the copy of V_δ in the first factor:

1. For all a, b, c, d, l as before, set

$$\begin{aligned} & \left[\sum_{\eta \in Z_2} l(\eta) \cdot Q(c+d+\eta) \cdot Q(c-d+\eta) \right] \cdot \left[\sum_{\eta \in Z_2} l(\eta) \cdot X_{a+b+\eta} \cdot X_{a-b+\eta} \right] - \\ & \quad - \left[\sum_{\eta \in Z_2} l(\eta) \cdot Q(c+b+\eta) \cdot Q(c-b+\eta) \right] \cdot \left[\sum_{\eta \in Z_2} l(\eta) \cdot X_{a+d+\eta} \cdot X_{a-d+\eta} \right] = 0. \end{aligned}$$

Thus via the projection p_2 , \bar{A}_δ is a projective scheme over \bar{M}_δ . Moreover, the diagonal Δ is a section

$$\varepsilon: \bar{M}_\delta \rightarrow \bar{A}_\delta$$

of the projective scheme $\bar{A}_\delta/\bar{M}_\delta$.

Further Theorem. *Let $A_\delta = p_1^{-1}(M_\delta)$. Then A_δ/M_δ with section Δ is an abelian scheme over M_δ and it is equal to the universal abelian scheme, with its identity section, embedded in $P(V_\delta) \times M_\delta$ by the morphism i defined earlier in this section.*

The remainder of this section will be devoted to proving these theorems.

Step 1. Without passing to any subset of \bar{M}_δ at all, $\bar{A}_\delta/\bar{M}_\delta$ has a kind of \mathfrak{g} -structure. In fact, let M be the sheaf induced on \bar{A}_δ by $\underline{O}_P(1)$. Then for all morphisms

$$f: T \rightarrow \bar{M}_\delta,$$

let M' be the induced sheaf on $\bar{A}_\delta \times_{\bar{M}_\delta} T$. We get a canonical homomorphism:

$$\{T\text{-valued points of } \underline{\mathcal{G}}(\delta)\} \rightarrow \left\{ \begin{array}{l} \text{group of } T\text{-automorphisms} \\ \lambda: \bar{A}_\delta \times_M T \rightarrow \bar{A}_\delta \times_M T, \\ \text{plus isomorphisms} \\ \varphi: M' \xrightarrow{\sim} \lambda^* M' \end{array} \right\}.$$

And whenever $\bar{A}_\delta \times_{\bar{M}_\delta} T$ is the subscheme of $P(V_\delta) \times T$ induced from an abelian scheme with δ -marking, these homomorphisms give us back the \mathfrak{g} -structure. To define this, assume for simplicity that T is connected, and that

$$(\alpha, x, l) \in \mathcal{G}^T(\delta),$$

$$x = (a_1, \dots, a_g),$$

$$l = (l_1, \dots, l_g).$$

Then we get a projective transformation μ of $P(V_\delta) \times T$ by the linear map

$$V_\delta \otimes \Gamma(T, \underline{O}_T) \rightarrow V_\delta \otimes \Gamma(T, \underline{O}_T),$$

$$f \mapsto f^*,$$

$$f^*((b_1, \dots, b_g)) = \alpha \cdot \prod l_i^{b_i} \cdot f(a_1 + b_1, \dots, a_g + b_g)$$

(regarding $V_\delta \otimes \Gamma(T, \underline{O}_T)$ as $\Gamma(T, \underline{O}_T)$ -valued functions on $K(\delta)$). In particular,

$$X_b \otimes 1 \rightarrow X_{b-x} \otimes (\alpha \cdot \prod l_i^{b_i - a_i}).$$

One checks immediately that the equations defining \bar{A}_δ are invariant under this substitution. Therefore the projective transformation μ re-

stricts to an automorphism λ of $\bar{A}_\delta \times_{\bar{M}_\delta} T$. But the linear map defining μ also defines an isomorphism

$$\psi: [\underline{O}_P(1) \otimes \underline{O}_T] \xrightarrow{\sim} \mu^* [\underline{O}_P(1) \otimes \underline{O}_T].$$

And ψ restricts on $\bar{A}_\delta \times_{\bar{M}_\delta} T$ to an isomorphism

$$\varphi: M' \xrightarrow{\sim} \lambda^* M'.$$

Step II. In the first step, we have used a little bit the actual structure of the equations defining \bar{M}_δ : now we shall use this structure in detail. We want to define first a canonical morphism

$$\bar{M}_{2\delta} \xrightarrow{\pi} \bar{M}_\delta$$

which will be used in Step V.

Let M (resp. M') be the invertible sheaf on \bar{M}_δ (resp. $\bar{M}_{2\delta}$) obtained by restricting $\underline{O}_P(1)$ from the ambient $P(V_\delta)$ (resp. $P(V_{2\delta})$). By definition, we shall have

$$\pi^*(M^2) \cong M'^2,$$

and we define π by its effect on the sections of the very ample sheaf M^2 :

$$(*) \quad \pi^*[Q(a+b) \cdot Q(a-b)] = \sum_{\eta \in Z_2} Q'(a+\eta) \cdot Q'(b+\eta),$$

where Q and Q' are the sections of M and M' defined above, and $a, b \in K(2\delta)$ satisfy $a+b \in K(\delta)$. To see that this defines a morphism π , we must check that when these values for π^* are substituted, the following are zero:

$$(1) \quad \begin{aligned} & \pi^*[Q(a) \cdot Q(b)] - \pi^*[Q(b) \cdot Q(a)] \\ & \pi^*[Q(a) \cdot Q(b)] \cdot \pi^*[Q(c) \cdot Q(d)] - \\ & \quad - \pi^*[Q(a) \cdot Q(d)] \cdot \pi^*[Q(c) \cdot Q(b)] \\ & \pi^*[Q(-a) \cdot Q(b)] - \pi^*[Q(a) \cdot Q(b)] \end{aligned}$$

for all $a, b, c, d \in K(\delta)$,

$$(2) \quad \begin{aligned} & \sum_{\eta \in Z_2} l(\eta) \cdot \pi^*[Q(a+b+\eta) \cdot Q(a-b+\eta)] \times \\ & \quad \times \sum_{\eta \in Z_2} l(\eta) \cdot \pi^*[Q(c+d+\eta) \cdot Q(c-d+\eta)] - \\ & \quad - \sum_{\eta \in Z_2} l(\eta) \cdot \pi^*[Q(a+d+\eta) \cdot Q(a-d+\eta)] \times \\ & \quad \times \sum_{\eta \in Z_2} l(\eta) \cdot \pi^*[Q(c+b+\eta) \cdot Q(c-b+\eta)] \end{aligned}$$

for all $a, b, c, d \in K(2\delta)$, $l \in Z_2$ such that a, b, c, d are congruent modulo $K(\delta)$.

The first 2 expressions in (1) are 0 by virtue of the relations imposed on the Q' 's; the last expression in (1) and the expression (2) are identically

zero by virtue of the commutativity and associativity of multiplication in the Q' 's.

It is clear from the definition of π that if L is an invertible sheaf of type δ on an abelian variety and we choose symmetric \mathfrak{g} -structures on (L, L^2) , then the corresponding geometric points x_1, x_2 in $\overline{M}_\delta, \overline{M}_{2\delta}$ are related by $\pi(x_2)=x_1$. The main result that we will need is:

Hardest Lemma. *Let a geometric point x in $\overline{M}_{2\delta}$ be the theta-null point assigned to some abelian variety with 2δ -marking. Then π is étale in a neighborhood of x .*

Proof. Let $y=\pi(x)$. By the infinitesimal criterion for a morphism to be étale ([3], § 3, Cor. 3.2), we must check the following: let A be an artin local ring with residue field k , the field of definition of x . Assume that $\sigma: \text{Spec}(A) \rightarrow \overline{M}_\delta$ is an A -valued point extending $y: \text{Spec}(k) \rightarrow \overline{M}_\delta$. Then there exists one and only one A -valued point $\tau: \text{Spec}(A) \rightarrow \overline{M}_{2\delta}$ such that $\sigma=\pi \circ \tau$, and such that τ extends x .

In down-to-earth language, suppose that at x , the coordinates $Q'(a)$ have values $\overline{q}'(a)$ in k ; and that at σ , the coordinates $Q(a)$ have values $q(a)$ in A . Then $q(a), \overline{q}'(a)$ satisfy the Riemann theta relation and symmetry, and if $\overline{q}(a)$ is the image of $q(a)$ in k , then $\overline{q}'(a), \overline{q}(a)$ are related by (*). We must show that there are elements $q'(a) \in A$ such that $q'(a)$ lifts $\overline{q}'(a)$, and is still a point of $\overline{M}_{2\delta}$ and such that $q'(a)$ and $q(a)$ are related by (*). We have one more thing to help us: the values $\overline{q}(a)$ and $\overline{q}'(a)$ come from an abelian variety.

The first thing to observe is that (*) nearly determines $q'(a)$: in fact, we get

$$\begin{aligned} (*)' \quad \sum_{\eta \in Z_2} l(\eta) \cdot q(a+b+\eta) \cdot q(a-b+\eta) \\ = \sum_{\eta \in Z_2} l(\eta) \cdot q'(a+\eta) \cdot \sum_{\eta \in Z_2} l(\eta) \cdot q'(b+\eta). \end{aligned}$$

Since the \overline{q}' 's come from an abelian variety and since $4|d_i$ for all i , we also have

For all $l \in Z_2$ and $a \in K(\delta)$, there is an element $b \in a + K(\delta)$ such that

$$\sum_{\eta \in Z_2} l(\eta) \cdot \overline{q}'(b+\eta) \neq 0$$

(cf. § 4, Proof of Theorem 1).

Set

$$\begin{aligned} U(a, b, l) &= \sum_{\eta \in Z_2} l(\eta) \cdot q(a+b+\eta) \cdot q(a-b+\eta), \\ x(a, l) &= \sum_{\eta \in Z_2} l(\eta) \cdot q'(a+\eta), \\ \bar{x}(a, l) &= \sum_{\eta \in Z_2} l(\eta) \cdot \overline{q}'(a+\eta). \end{aligned}$$

For each $l \in Z_2$, and $\alpha \in K(2\delta)/K(\delta)$, choose an element $a_0 \in K(2\delta)$ lifting α such that $\bar{x}(a_0, l) \neq 0$. Then $U(a_0, a_0, l)$ is a unit in A , and $x(a_0, l)$ is to satisfy:

$$\text{i) } x(a_0, l)^2 = U(a_0, a_0, l), \quad x(a_0, l) \mapsto \bar{x}(a_0, l) \text{ in } k.$$

Since $\text{char}(k) \neq 2$, this determines one and only one $x(a_0, l)$. For any other $a \in K(2\delta)$ lifting α , set

$$\text{ii) } x(a, l) = \frac{U(a, a_0, l)}{x(a_0, l)}.$$

From the x 's, we determine the q 's by summing over l . This proves the uniqueness of the values $q'(a)$, i.e., of the point τ , and shows that π is unramified at x . It remains to show that if $\{x(a, l)\}$, hence $\{q'(a)\}$ are determined by i), ii), then they satisfy all the requirements.

First of all, $x(a, l)$ lifts $\bar{x}(a, l)$ since $\bar{x}(a, l)$ satisfies (ii) with bars in it. Hence $q'(a)$ lifts $\bar{q}'(a)$.

Secondly, $x(a, l)$ and $U(a, b, l)$ satisfy (*): let $a, b \in K(2\delta)$ both lie over α . Then

$$\begin{aligned} x(a, l) \cdot x(b, l) &= \frac{U(a, a_0, l) \cdot U(b, a_0, l)}{x(a_0, l)^2} \\ &= \frac{U(a, a_0, l) \cdot U(b, a_0, l)}{U(a_0, a_0, l)} \\ &= U(a, b, l) \end{aligned}$$

by RIEMANN's theta relation for the q 's. Therefore, the corresponding $q'(a)$'s satisfy (*).

Thirdly, q' is even. In fact, I claim $x(-a, l) = x(a, l)$. This follows from the equation $U(-a, a_0, l) = U(a, a_0, l)$ which follows from the evenness of q .

Fourthly, we must check that q' satisfies RIEMANN's theta relation. This is really rather tricky. It is convenient to use a different form of these relations:

$$\begin{aligned} \text{let } H &= K(2\delta) \times \hat{Z}_4 \\ [n.b.: \hat{Z}_4 \text{ is the discrete group } \text{Hom}(Z_4, A^*)]. \end{aligned}$$

If $b_i = (a_i, l_i) \in H$, $i = 1, 2$, then set

$$\begin{aligned} C(b_1, b_2) &= \sum_{\substack{\eta_1, \eta_2 \in Z_4 \\ \eta_1 + \eta_2 \in Z_2}} l_1(\eta_1) l_2(\eta_2) q'(a_1 + \eta_1) \cdot q'(a_2 + \eta_2) \\ &= \frac{1}{2^g} \sum_{\eta \in Z_4} (l_1 + l_2)(\eta) \cdot x(a_1 + \eta, \bar{l}_1) \cdot x(a_2 + \eta, \bar{l}_2) \end{aligned}$$

[here \bar{l}_i = restrictions of l_i to Z_2].

The relations are:

$$\begin{aligned}
 C(b_1, b_2) \cdot C(b_3, b_4) &= C(b_1 + \beta, b_2 + \beta) \cdot C(b_3 + \beta, b_4 + \beta) \\
 (**) \quad &\text{for all } b_1, b_2, b_3, b_4, \beta \in H \text{ such that} \\
 &b_1 + b_2 + b_3 + b_4 = -2\beta.
 \end{aligned}$$

To check that these relations are equivalent to the ones we have been using involving U 's is quite straightforward and we omit it. [The method is the same as the one we will use in the next step.]

We also need a *third* form of these relations, that is, the Riemann form itself involving the x 's (cf. § 3, right at the end):

$$\begin{aligned}
 &x(a_1, l_1) \cdot x(a_2, l_2) \cdot x(a_3, l_3) \cdot x(a_4, l_4) \\
 &= \frac{1}{2^{2g}} \sum_{\substack{k \in \hat{Z}_2 \\ \eta \in Z_4}} k(2\eta) \cdot x(a_1 + b + \eta, l_1 + k) \cdot x(a_2 + b + \eta, l_2 + k) \times \\
 &\quad \times x(a_3 + b + \eta, l_3 + k) \cdot x(a_4 + b + \eta, l_4 + k) \\
 (***) \quad &\text{for all } a_1, a_2, a_3, a_4, b \in K(2\delta) \\
 &\quad l_1, l_2, l_3, l_4 \in \hat{Z}_2 \\
 &\text{such that} \\
 &\quad a_1 + a_2 + a_3 + a_4 = -2b, \\
 &\quad l_1 + l_2 + l_3 + l_4 = 0.
 \end{aligned}$$

This is how to go back and forth between (**) and (***):

1) take expression (**), substitute $l_1 + 2k_1$ for l_1 , $l_3 + 2k_3$ for l_3 , and $m - k_1 - k_3$ for m , [here $b_i = (a_i, l_i)$, and $\beta = (\alpha, m)$]. Sum over all choices of k_1, k_3 , rearrange the right-hand side and you get (***).

2) take expression (***), substitute $a_1 + \zeta$ for a_1 , $a_2 + \zeta$ for a_2 , $a_3 + \zeta'$ for a_3 , $a_4 + \zeta'$ for a_4 , $b - \zeta - \zeta'$ for b , multiply by $(l_1 + l_2)(\zeta) \cdot (l_3 + l_4)(\zeta')$, and sum over all $\zeta, \zeta' \in Z_4$. Rearrange the right-hand side and you get (**).

The point to notice here is that you don't need all of the relations (**) to get a particular one of the relations (***), or vice versa.

Next, there are some relations (**) that we do get immediately. In fact, suppose $b_1 + b_2 \in 2H$. Then $a_1 + a_2 \in K(\delta)$ and $l_1 = l_2$ on \hat{Z}_2 . Using (*), one finds that $C(b_1, b_2)$ splits into a product:

$$\begin{aligned}
 C(b_1, b_2) &= \frac{1}{2^{2g}} \left[\sum_{\zeta \in Z_4} (l_1 + l_2)(\zeta) \cdot q(a_1 + a_2 + 2\zeta) \right] \times \\
 &\quad \times \left[\sum_{\zeta \in Z_4} (l_1 - l_2)(\zeta) \cdot q(a_1 - a_2 + 2\zeta) \right].
 \end{aligned}$$

One gets similar equations for $C(b_3, b_4)$, $C(b_1 + \beta, b_2 + \beta)$ and $C(b_3 + \beta, b_4 + \beta)$: substituting, one gets (**) from the associative law.

Next, we can show that the relations (**) also hold if $b_1 + b_3 \in 2H$. We prove this in 3 steps using the precise description of which relations (**) are needed to prove which relations (***) and vice versa.

i) relations (**) for all values of $b_1, b_2, b_3, b_4, \beta$ such that $b_1 + b_2 \in 2H$ imply relations (***) for all a_i, l_i, b such that $a_1 + a_2 \in K(\delta)$, $l_1 = l_2$ on Z_2 .

ii) relations (***) are symmetric under permutation of the variables. So we also get relations (***) when $a_1 + a_3 \in K(\delta)$, $l_1 = l_3$ on Z_2 .

iii) Using the fact that $Z_4 \subset K(\delta)$, if $a_1 + a_3 \in K(\delta)$, then $a_1 + a_3 + \zeta + \zeta' \in K(\delta)$ and we can go back: we get relations (**) whenever $b_1 + b_3 \in 2H$.

Next, we can show that the square of relation (**) is always true:

$$C(b_1, b_2)^2 \cdot C(b_3, b_4)^2 = C(b_1 + \beta, b_2 + \beta)^2 \cdot C(b_3 + \beta, b_4 + \beta)^2.$$

In fact, for all $b_1, b_2 \in H$,

$$\begin{aligned} C(b_1, b_2)^2 &= C(b_1, b_2) \cdot C(b_1, -b_2) \\ &= C(0, b_2 - b_1) \cdot C(0, -b_1 - b_2) \end{aligned}$$

by relation (**) with $\beta = -b_1$

$$= C(b_1 - b_2, 0) \cdot C(b_1 + b_2, 0).$$

Therefore:

$$\begin{aligned} C(b_1 + \beta, b_2 + \beta)^2 \cdot C(b_3 + \beta, b_4 + \beta)^2 \\ = C(b_1 + b_2 + 2\beta, 0) \cdot C(b_1 - b_2, 0) \cdot C(b_3 + b_4 + 2\beta, 0) \times \\ \times C(b_3 - b_4, 0). \end{aligned}$$

But

$$\begin{aligned} b_1 + b_2 + 2\beta &= -b_3 - b_4, \quad \text{and} \quad b_3 + b_4 + 2\beta = -b_1 - b_2: \quad \text{so we get} \\ &= C(b_1 + b_2, 0) \cdot C(b_1 - b_2, 0) \cdot C(b_3 + b_4, 0) \cdot C(b_3 - b_4, 0) \\ &= C(b_1, b_2)^2 \cdot C(b_3, b_4)^2. \end{aligned}$$

But the Riemann relations definitely hold when you take the image of $C(b_1, b_2)$, etc., in k . Therefore, if

$$\bar{C}(b_1, b_2) \cdot \bar{C}(b_3, b_4) \neq 0 \quad \text{in } k,$$

it follows that $C(b_1, b_2) \cdot C(b_3, b_4)$ and $C(b_1 + \beta, b_2 + \beta) \cdot C(b_3 + \beta, b_4 + \beta)$ are square roots of the same number, are units, and have the same images in k : hence they are equal.

Now suppose

$$b_1 + b_2 + b_3 + b_4 = -2\beta$$

$$b'_1 + b'_2 + b'_3 + b'_4 = -2\beta'$$

where $b_i + b'_i \in 2H$, all i , and $\beta + \beta' \in 2H$. Assume moreover that

$$C(b'_1, b'_2) \cdot C(b'_3, b'_4) \quad \text{is a unit.}$$

Then we can also show that

$$C(b_1, b_2) \cdot C(b_3, b_4) = C(b_1 + \beta, b_2 + \beta) \cdot C(b_3 + \beta, b_4 + \beta).$$

In fact, we know this equality, when there are primes everywhere. Hence it suffices to show:

$$\begin{aligned} & C(b_1, b_2) \cdot C(b'_1, b'_2) \cdot C(b_3, b_4) \cdot C(b'_3, b'_4) \\ (\alpha) \quad &= C(b_1 + \beta, b_2 + \beta) \cdot C(b'_1 + \beta', b'_2 + \beta') \cdot C(b_3 + \beta, b_4 + \beta) \times \\ & \quad \times C(b'_3 + \beta', b'_4 + \beta'). \end{aligned}$$

But assume $b_i + b'_i = -2\gamma_i$. Then using the relations (**) that we know to be true, we get:

$$\begin{aligned} & C(b_1, b_2) \cdot C(b'_1, -b'_2) \\ &= C(\gamma_1 - \gamma_2 + b_1 - b_2, \gamma_1 - \gamma_2) \cdot C(\gamma_1 + \gamma_2 + b'_1 + b'_2, \gamma_1 + \gamma_2) \\ & C(b_3, b_4) \cdot C(b'_3, -b'_4) \\ &= C(\gamma_3 - \gamma_4 + b_3 - b_4, \gamma_3 - \gamma_4) \cdot C(\gamma_3 + \gamma_4 + b'_3 + b'_4, \gamma_3 + \gamma_4) \\ & C(b_1 + \beta, b_2 + \beta) \cdot C(b'_1 + \beta', -b'_2 - \beta') \\ &= C(b_1 - b_2 + \gamma_1 - \gamma_2, \gamma_1 - \gamma_2) \times \\ & \quad \times C(\gamma_1 + \gamma_2 + b'_1 + b'_2 + \beta' - \beta, \gamma_1 + \gamma_2 - \beta - \beta') \\ & C(b_3 + \beta, b_4 + \beta) \cdot C(b'_3 + \beta', -b'_4 - \beta') \\ &= C(b_3 - b_4 + \gamma_3 - \gamma_4, \gamma_3 - \gamma_4) \times \\ & \quad \times C(\gamma_3 + \gamma_4 + b'_3 + b'_4 + \beta' - \beta, \gamma_3 + \gamma_4 - \beta - \beta'). \end{aligned}$$

We can assume that $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = \beta + \beta'$ by altering one of the γ 's by an element of order 2. Using the symmetries $C(A, B) = C(A, -B) = C(B, A)$ one checks that the product of the 1st two right-hand sides equals the product of the 2nd two. Hence the same for the left-hand sides, hence (α) is true.

It remains only to check that for *any* $b_1, b_2, b_3, b_4, \beta$ such that $b_1 + b_2 + b_3 + b_4 = -2\beta$, there exists $b'_1, b'_2, b'_3, b'_4, \beta'$ differing from the b_i 's and β by $2H$, such that

$$\begin{aligned} b'_1 + b'_2 + b'_3 + b'_4 &= -2\beta' \\ \bar{C}(b'_1, b'_2) \cdot \bar{C}(b'_3, b'_4) &\neq 0. \end{aligned}$$

Now we know that the functions \bar{q}, \bar{q}' come from an abelian variety. Therefore choosing a compatible 4δ -marking on this abelian variety, there is a null-value function \bar{q}'' on $K(4\delta)$ with values in k , such that

$$\bar{q}'(a+b) \cdot \bar{q}'(a-b) = \sum_{\eta \in Z_2} \bar{q}''(a+\eta) \cdot \bar{q}''(b+\eta).$$

Now if $a \in K(4\delta)$, $l \in \hat{Z}_4$, set:

$$\bar{z}(a, l) = \sum_{\eta \in Z_4} l(\eta) \bar{q}''(a+\eta).$$

Then if $b_1 = (a+b, l)$, $b_2 = (a-b, m)$ are 2 elements of H , with $a, b \in K(4\delta)$, $l, m \in \hat{Z}_4$, one checks immediately that:

$$\bar{C}(b_1, b_2) = \bar{z}(a, l+m) \cdot \bar{z}(b, l-m).$$

Therefore, translating the question into one involving \bar{z} 's, we have to check:

$$\left\{ \begin{array}{l} \text{for all } a_1, a_2, a_3, a_4 \in K(4\delta), \quad \text{congruent mod } K(2\delta), \\ \text{for all } l_1, l_2, l_3, l_4 \in \hat{Z}_4, \quad \text{congruent mod } 2\hat{Z}_4, \\ \text{there exists } \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in K(2\delta), \text{ congruent mod } K(\delta) \\ \text{and } k \in 2\hat{Z}_4, \text{ such that} \\ \prod_{i=1}^4 \bar{z}(a_i + \alpha_i, l_i + m) \neq 0. \end{array} \right.$$

But using the hypothesis that $Z_8 \subset K(\delta)$, this follows from:

$$\left\{ \begin{array}{l} \text{for all } a \in K(4\delta), l \in \hat{Z}_4, \text{ there exists } \alpha \in Z_8 \text{ such that} \\ \bar{z}(a + \alpha, l) \neq 0. \end{array} \right.$$

This is a special case of the result stated following Theorem 1, § 4. Phew!

The same analysis also gives the result:

If $\{\bar{q}'(a)_1\}$ are the coordinates of a geometric point x_1 in $\bar{M}_{2\delta}$ which corresponds to an abelian variety with 2δ -marking, then the coordinates $\{\bar{q}'(a)_2\}$ of any other geometric point x_2 such that $\pi(x_1) = \pi(x_2)$ are determined by:

$$\sum_{\eta \in Z_2} l(\eta) \bar{q}'(a+\eta)_2 = \gamma(a, l) \cdot \sum_{\eta \in Z_2} l(\eta) \cdot \bar{q}'(a+\eta)_1$$

where γ is a quadratic character:

$$\gamma: K(2\delta)/K(\delta) \times \mathbb{Z}_2 \rightarrow \{\pm 1\}.$$

We leave it to the reader to check this change from the $\bar{q}'(a)_1$'s to the $\bar{q}'(a)_2$'s also comes about by a suitable modification of the 2δ -marking. Therefore:

if a geometric point x of $\bar{M}_{2\delta}$ corresponds to an abelian variety with 2δ -marking, so do all geometric points in $\pi^{-1}(\pi(x))$.

Step III. Let \mathcal{X}/S be an abelian scheme with δ -markings. Let $i: \mathcal{X} \hookrightarrow P(V_\delta) \times S$ be the closed immersion defined above. Then

- i) $p_1 \circ i \circ \varepsilon$ is actually a morphism j of S to \bar{M}_δ ,
- ii) $i(\mathcal{X})$ equals the subscheme \mathcal{Y} of $P(V_\delta) \times S$ obtained as the fibre product:

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & P(V_\delta) \times S \\ \downarrow & & \downarrow id \times j \\ \bar{A}_\delta & \longrightarrow & P(V_\delta) \times \bar{M}_\delta. \end{array}$$

In order to establish this, it clearly suffices to take the case $S = \text{Spec}(A)$, where A is an artin local ring with algebraically closed residue field. Let p be the characteristic of the residue field k . Then there are considerable technical simplifications:

- a) for all positive integers n not divisible by p , A contains exactly n n -th roots of 1, which lift the n n -th roots of 1 in k ,
- b) if \mathcal{X}/S is an abelian scheme, then for all positive integers n not divisible by p , the subscheme \mathcal{X}_n of points of order n is the disjoint union of n^{2g} subschemes, each being the image of a section of \mathcal{X}/S : i.e., all points of order n are rational over S .

Cor. of (a), (b). Let \mathcal{X}/S be an abelian scheme, and let X/k be its fibre over the residue field. Let L be a relatively ample sheaf of degree d on \mathcal{X} . Then $H(L)$ is a disjoint union of the images of d^2 distinct sections of \mathcal{X}/S , and there is a one-one correspondence between \mathfrak{g} -structures for L and for the induced sheaf \bar{L} on X .

In this case, we can deal entirely with the group of rational points $\mathcal{G}^S(\delta)$ instead of with the combersome group schemes. Thus $\mathcal{G}^S(\delta)$ is just the (discrete) group extension:

$$0 \rightarrow A^* \rightarrow \mathcal{G}^S(\delta) \rightarrow K(\delta) \times \widehat{K(\delta)} \rightarrow 0$$

and where $\widehat{K(\delta)}$ is just a (discrete) group isomorphic to $K(\delta)$. And the representation theory boils down to:

There is a unique A -linear representation of $\mathcal{G}^S(\delta)$ on a free A -module of rank d , in which the subgroup A^* acts by homotheties.

In particular, a \mathfrak{g} -structure on L induces an isomorphism:

$$\Gamma(\mathcal{X}, L) \cong V_\delta \otimes_{\mathbf{Z}} A$$

unique up to multiplication by elements of A^* .

To get down to the proof itself, we are given \mathcal{X}/S , plus a totally symmetric relatively ample L , plus a symmetric isomorphism f_1 of $\mathcal{G}(L)$ and $\mathcal{G}^S(\delta)$. By the results of § 2, there is a \mathfrak{g} -structure

$$\bar{f}_2: \mathcal{G}(\bar{L}^2) \xrightarrow{\sim} \mathcal{G}^S(2\delta)$$

such that the pair \bar{f}_1 and \bar{f}_2 of \mathfrak{g} -structures on X is symmetric. Lift \bar{f}_2 to a \mathfrak{g} -structure f_2 for L^2 . Then f_1 and f_2 induce isomorphisms:

$$\Gamma(\mathcal{X}, L) \cong V_\delta \otimes_{\mathbf{Z}} A$$

$$\Gamma(\mathcal{X}, L^2) \cong V_{2\delta} \otimes_{\mathbf{Z}} A,$$

where the first is the isomorphism used to define $i: \mathcal{X} \hookrightarrow \mathbf{P}(V_\delta) \times S$. Also, evaluation of sections at ε defines particular A -valued functions q_L and q_{L^2} on $K(\delta)$ and $K(2\delta)$ via

$$\begin{array}{ccc} \Gamma(\mathcal{X}, L^i) \cong V_{i\delta} \otimes_{\mathbf{Z}} A & & \\ \downarrow \varepsilon^* & \downarrow f & \mapsto \sum_{a \in K(i\delta)} f(a) q_{L^i}(a). \\ A & = & A \end{array}$$

Here $q_L(a)$ is just the value of $Q(a)$ at ε : so $\{q_L(a)\}$ is a set of homogeneous coordinates for the point $i \circ \varepsilon: S \rightarrow \mathbf{P}(V_\delta) \times S$. This means that $\{q_L(a)\}$ are the homogeneous coordinates of $j(S)$, and these are exactly the numbers which are to satisfy RIEMANN's theta relations.

The only thing that must really be checked is that the pairing

$$\Gamma(\mathcal{X}, L) \times \Gamma(\mathcal{X}, L) \rightarrow \Gamma(\mathcal{X}, L^2)$$

given by multiplying sections is defined by the same multiplication formula as in § 3. But the proof given in § 3 of this formula goes over without any change to this more general case: simply replace the ground field k by the ground ring A . Once the multiplication formula is known, RIEMANN's relations on $\{q_L(a)\}$ follow formally, as in § 3. This proves (i).

To prove (ii), the first point is that $\mathcal{X} \subset \mathcal{Y}$. This follows immediately from the multiplication formula. In fact, on \mathcal{X} , the coordinates $Q(a)$ have the constant values $q_L(a)$ and the coordinates X_a induce the sections of L on \mathcal{X} denoted by δ_a previously. The identity which we need comes out immediately by reducing the quadratic expressions in the δ_a 's to

functions on $K(2\delta)$ and using the relation between q_L and q_{L^2} . Secondly, if Y is the fibre of \mathcal{Y} over k , then $X=Y$. This follows from Theorem 2, § 4, and the fact that the equations on X_a are a complete set of quadratic relations. In fact, for fixed $l \in Z_2$, $a \in K(2\delta)$, these relations assert that all the quadratic expressions

$$\left\{ \sum_{\eta \in Z_1} l(\eta) X_{a+b+\eta} \cdot X_{a-b+\eta} \mid b \in a + K(\delta) \right\}$$

are proportional. Since, in fact, on X each of these sets consists in sections of L^2 corresponding to the function $Y_{a,l}$:

$$Y_{a,l}(b) = \begin{cases} 0 & \text{if } b-a \notin Z_2 \\ l(b-a) & \text{if } b-a \in Z_2, \end{cases}$$

and since the $Y_{a,l}$'s are linearly *independent*, these do exhaust the quadratic relations on the X_a . Thirdly, we note the obvious:

Lemma. *Let U be a closed subscheme of V , both being noetherian schemes over $\text{Spec}(A)$. Then if*

1. *U is flat over A ,*
2. *the fibres \bar{U} and \bar{V} over $\text{Spec}(k)$ are equal, then $U=V$.*

This proves (ii).

Step IV. Suppose that N is a connected subscheme of \bar{M}_δ such that if \bar{A}_δ induces a subscheme B over N , then B is smooth over N . Suppose that some geometric point x of N is in the image of \mathcal{M}_δ under t . Then the N -valued point 1_N of N is in the image of \mathcal{M}_δ , i.e., there is an abelian scheme \mathcal{Y}/N with δ -marking such that $i(\mathcal{Y})=B$.

By assumption, one geometric fibre \bar{B} of B/N is an abelian variety. Therefore, by Theorem 6.14, Ch. 6, [9], B is an abelian scheme over N with identity section Δ . By construction, $B \subset P(V_\delta) \times N$. Let $\underline{O}_P(1)$ induce the sheaf M on B . Normalize it on the identity section, i.e., set

$$L = M \otimes p_2^* \Delta^*(M^{-1}).$$

I claim that the inverse ι for B is given by restricting the projective transformation j to B , where

$$j: P(V_\delta) \times N \rightarrow P(V_\delta) \times N$$

$$j^*(X_a) = X_{-a}.$$

But the inverse is given by j on \bar{B} (cf. § 3). Since $\iota=j$ on one geometric fibre of B/N and on the section Δ of B/N , $\iota=j$ everywhere by the rigidity lemma ([9], Ch. 6, §1). Since j is a projective transformation, this shows that $\iota^*L \cong L$, i.e., L is symmetric. Now j has 2 disjoint subspaces of fixed

points:

$$L_1 \text{ defined by } X_a = X_{-a}, \quad \text{all } a \in K(\delta)$$

$$L_2 \text{ defined by } X_a = -X_{-a}, \quad \text{all } a \in K(\delta)$$

and the identity section Δ is completely contained in L_1 . To say that L is totally symmetric is the same as saying that B_2 , the kernel of multiplication by 2, is completely contained in L_1 . But $B_2 \subset L_1 \cup L_2$ since $j=id$ on B_2 ; and \bar{B}_2 , the points of order 2 on \bar{B} , is contained in L_1 since the projective embedding of \bar{B} is assumed to come from a δ -marking. Since S is connected, this implies that $B_2 \subset L_1$, and hence that L is totally symmetric.

Moreover, by Step I, we get a homomorphism

$$\lambda: \mathcal{G}(\delta) \times N \rightarrow \mathcal{G}(L)$$

of group schemes over N such that the actions on $p_{2,*}L$ and V_δ match up. Since λ induces an isomorphism of the group schemes $\mathcal{G}(\delta) \times \text{Spec}(k)$, $\mathcal{G}(\bar{L})$ corresponding to the situation in the geometric fibre \bar{B} , λ is an isomorphism everywhere². Therefore, λ is a \mathfrak{g} -structure. Also λ is symmetric at one point, hence it is symmetric everywhere. Therefore, B/N has a \mathfrak{g} -marking which induces the given embedding in $P(V_\delta) \times N$. *Q. E. D.*

Step V. Recall the main result on flattening stratifications ([10], Lecture 8):

Given a projective morphism $f: X \rightarrow Y$ of noetherian schemes, Y can be decomposed into a disjoint union of locally closed connected subschemes Y_α , such that if $g: Y' \rightarrow Y$ is a morphism from a connected noetherian scheme Y' to Y , and we look at the fibre product:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y, \end{array}$$

then f' is flat if and only if g factors through one of the subschemes Y_α .

Apply this to $\bar{A}_\delta/\bar{M}_\delta$: if $Z \subset \bar{M}_\delta$ is one of the pieces, and some fibre of \bar{A}_δ over Z is obtained from an abelian scheme with δ -marking, then we must show that Z is *open*. If we do this, both theorems are proven. The openness of Z follows from:

Key Lemma. *Let $X/\text{Spec}(k)$ be an abelian variety with δ -marking over an algebraically closed field k . Let this define a geometric point x of \bar{M}_δ .*

² In fact, since λ is an isomorphism of the subgroups $G_{m,S}$ in any case, it is only a question of whether λ is an isomorphism of the quotients $\bar{H}(\delta)$, $\bar{H}(L)$. Since these are etale over S , a homomorphism is an isomorphism if it is so at one point.

Let A be any artin local ring with residue field k , and let

$$f: \operatorname{Spec}(A) \rightarrow \overline{M}_g$$

be a morphism extending x . Then the scheme over $\operatorname{Spec}(A)$ induced by \overline{A}_g is flat over $\operatorname{Spec}(A)$.

Proof of Lemma. Let \overline{A}_g define, over A , the projective scheme

$$\mathcal{X} \subset \mathbf{P}(V_g) \times \operatorname{Spec}(A).$$

Let L be the invertible sheaf on \mathcal{X} obtained by restricting $\mathcal{O}_{\mathbf{P}}(1)$ to \mathcal{X} . We shall show:

$$(*) \quad \Gamma(\mathcal{X}, L^{2^n}) \quad \text{is a free } A\text{-module, for all large } n.$$

This certainly is enough to prove the lemma, in view of the following elementary observation:

Sublemma. Let A be an artin local ring with residue field k , and let R be a graded A -algebra generated over $A = R_0$ by R_1 . Assume that R_1 is a finite A -module, and that R_n is a free A -module for an infinite set of integers n . Then R_n is free for all but a finite set of n 's.

Proof. Without loss of generality, we can assume that k is infinite. Let $\overline{R} = R \otimes_A k$. In \overline{R} , let

$$(0) = Q_1 \cap \cdots \cap Q_l, \quad P_i = \sqrt{Q_i}$$

be the primary decomposition of (0) . Let P_1 be the irrelevant ideal

$$\bigoplus_{n \geq 1} \overline{R}_n.$$

Since k is infinite, there is an element $g \in R_1$ such that

$$\overline{g} \notin P_2 \cup \cdots \cup P_l.$$

Let n_0 be an integer such that $Q_1 \supset P_1^{n_0}$, i.e., $Q_1 \supset R_n$ for all $n \geq n_0$. I claim that R_m is a free A -module for any integer $m \geq n_0$. Choose f_1, \dots, f_k in R_m such that $\overline{f}_1, \dots, \overline{f}_k$ are a basis of \overline{R}_m . Note that for all integers p ,

$$\overline{g}^p \overline{f}_1, \dots, \overline{g}^p \overline{f}_k \in \overline{R}_{m+p}$$

are independent over k . In fact, if

$$0 = \sum \alpha_i \overline{g}^p \overline{f}_i = \overline{g}^p \cdot (\sum \alpha_i \overline{f}_i)$$

then $\overline{g} \notin P_i$, for $i \geq 2$ implies $\sum \alpha_i \overline{f}_i \in Q_i$, $i \geq 2$. And since the degree is large enough, $\sum \alpha_i \overline{f}_i \in Q_1$. Hence $\sum \alpha_i \overline{f}_i = 0$, which is a contradiction

unless all the α 's are 0. Now suppose that R_m is not a free A -module. Then

$$\sum_{i=1}^k a_i f_i = 0$$

for some elements $a_1, \dots, a_k \in A$, not all 0. Let p be a positive integer such that R_{m+p} is a free A -module. Then since \bar{R}_{m+p} has a basis of the form $\bar{g}^p \bar{f}_1, \dots, \bar{g}^p \bar{f}_k, \bar{h}_1, \dots, \bar{h}_l$, it follows that R_{m+p} has a basis of the form $g^p f_1, \dots, g^p f_k, h_1, \dots, h_l$. But

$$\sum_{i=1}^k a_i (f_i g^p) = 0.$$

Therefore, the a 's are all 0 which is a contradiction. *Q.E.D.*

Once we know that $\Gamma(\mathcal{X}, L^n)$ is a free A -module for all large n , then \mathcal{X} is flat over $\text{Spec}(A)$ ([5], Ch. 3, 7.9.14). To prove (*), let $I_\delta \subset S^*(V_\delta) \otimes A$ be the ideal generated by the quadratic defining relations of \bar{A}_δ . Then

$$\Gamma(\mathcal{X}, L^n) \cong (S^n V_\delta \otimes A) / (I_\delta)_n$$

for all large n , and it suffices to prove:

$$(**) \quad (I_\delta)_{2^n} \text{ is a direct summand of } (S^{2^n} V_\delta) \otimes A \\ \text{for all } n=0, 1, 2, \dots$$

We shall prove (**) by induction on n , assuming at each stage that it is known for $n \leq n_0$ and all δ . To start things off, $(I_\delta)_1$ equals (0), for (**) is always true for $n=0$. To get from (**) for $n=n_0$ to (**) for $n=n_0+1$, we shall simply show for all A -valued points of \bar{M}_δ as above, there is an A -valued point of $\bar{M}_{2\delta}$ with the same properties such that:

$$(***) \quad \bigoplus_{n=0}^{\infty} ((S^{2^n} V_\delta) \otimes A) / (I_\delta)_{2^n} \cong ((S^* V_{2\delta}) \otimes A) / I_{2\delta}.$$

In particular,

$$((S^{2^{n+1}} V_\delta) \otimes A) / (I_\delta)_{2^{n+1}} \cong (S^{2^n} V_{2\delta} \otimes A) / (I_{2\delta})_{2^n}$$

so (**) will be completely proven.

To lift the A -valued point of \bar{M}_δ , we use the morphism

$$\pi: \bar{M}_{2\delta} \rightarrow \bar{M}_\delta$$

defined in Step II. The underlying k -valued point of f corresponds to an abelian variety with δ -marking. Choose a compatible 2δ -marking on this abelian variety. This defines a k -valued point y of $\bar{M}_{2\delta}$ over x . By Step II, π is étale at x . Therefore, there is one (and only one) A -valued point

$$g: \text{Spec}(A) \rightarrow \bar{M}_{2\delta}$$

lifting f with underlying k -valued point y .

Let these A -valued points be given by homogeneous coordinates $q(a)$, $a \in K(\delta)$, and $q'(b)$, $b \in K(2\delta)$, where $q(a)$, $q'(b) \in A$. These satisfy the usual equations (cf. Step II). Then in concrete terms:

$$S^* V_\delta \otimes A/I_\delta \cong \frac{A[\dots, X_a, \dots]_{a \in K(\delta)}}{\left\{ \begin{aligned} &(\Sigma l(\eta) q(c+d+\eta) \cdot q(c-d+\eta)) \cdot (\Sigma l(\eta) X_{a+b+\eta} X_{a-b+\eta}) \\ &-(\Sigma l(\eta) q(c+b+\eta) \cdot q(c-b+\eta)) \cdot (\Sigma l(\eta) X_{a+d+\eta} X_{a-d+\eta}) \end{aligned} \right\}}$$

$$S^* V_{2\delta} \otimes A/I_{2\delta} \cong \frac{A[\dots, X'_a, \dots]_{a \in K(2\delta)}}{\left\{ \begin{aligned} &(\Sigma l(\eta) q'(c+d+\eta) \cdot q'(c-d+\eta)) \cdot (\Sigma l(\eta) X'_{a+b+\eta} X'_{a-b+\eta}) \\ &-(\Sigma l(\eta) q'(c+b+\eta) \cdot q'(c-b+\eta)) \cdot (\Sigma l(\eta) X'_{a+d+\eta} X'_{a-d+\eta}) \end{aligned} \right\}}.$$

We set up the isomorphism (***) by requiring that

$$T[X_{a+b} \cdot X_{a-b}] = \sum_{\eta \in Z_2} q'(b+\eta) \cdot X'_{a+\eta}$$

for all $a, b \in K(2\delta)$ such that $a+b \in K(\delta)$. This implies:

$$T\left[\sum_{\eta \in Z_2} l(\eta) X_{a+b+\eta} X_{a-b+\eta}\right] = \sum_{\eta \in Z_2} l(\eta) q'(b+\eta) \cdot \sum_{\eta \in Z_2} l(\eta) X'_{a+\eta}$$

for all $a, b \in K(2\delta)$, $l \in \hat{Z}_2$ such that $a+b \in K(\delta)$. Since

$$\Sigma l(\eta) q(a+b+\eta) q(a-b+\eta) = \Sigma l(\eta) q'(a+\eta) \cdot \Sigma l(\eta) q'(b+\eta),$$

this second equation implies that the quadratic relations in the ring of X 's go to 0 in the ring of X' 's. Moreover, since for all $a \in K(2\delta)$, $l \in \hat{Z}_2$, there is an element $b \in a + K(\delta)$ such that $\sum (l(\eta) q'(b+\eta))$ is a *unit*, this also shows that the map is surjective. Similarly, since the elements $\sum l(\eta) X'_{a+\eta}$ are linearly independent it is clear from this second equation that every quadratic expression in the X 's which goes to 0 via T is a combination of expressions of the form:

$$\sum_{b \in a + K(\delta)} A_b \left\{ \sum_{\eta \in Z_2} l(\eta) X_{a+b+\eta} X_{a-b+\eta} \right\}.$$

And, in fact, the expressions

$$\begin{aligned} &[\Sigma l(\eta) q'(b_2+\eta)] \cdot [\Sigma l(\eta) X_{a+b_1+\eta} \cdot X_{a-b_1+\eta}] \\ &- [\Sigma l(\eta) q'(b_1+\eta)] \cdot [\Sigma l(\eta) X_{a+b_2+\eta} X_{a-b_2+\eta}] \end{aligned}$$

will span the kernel. But if $b_1 = b$, $b_2 = d$ and we multiply this expression by a *unit* of the form $\sum l(\eta) q'(c+\eta)$, $c \in a + K(\delta)$, we get the typical quadratic relation on the X 's. Therefore, T sets up an isomorphism:

$$S^2 V_\delta \otimes A/(I_\delta)_2 \xrightarrow{\sim} S^1 V_{2\delta} \otimes A/(I_{2\delta})_1.$$

At this stage in the proof, we have already shown that for *all* δ , $(I_\delta)_2$ is a direct summand of $S^2 V_\delta \otimes A$.

Now try to extend T to a homomorphism of the whole ring. Let U be the free A -module $S^2 V_\delta \otimes A/(I_\delta)_2$. Let $Y_{a,b}$ be the element $X_a X_b$ in U . Then, as observed in the proof of Theorem 2, § IV,

$$\bigoplus_{n=0}^{\infty} S^{2n} V_\delta \otimes A/(I_\delta)_{2n} \cong S^* U / \left\{ \text{ideal generated by } \begin{matrix} Y_{a,b} Y_{c,d} - Y_{a,d} Y_{c,b} \end{matrix} \right\}.$$

Therefore, all we have to check is that when T is extended to $S^2 U$, the A -module generated by the expressions $T(Y_{a,b}) \cdot T(Y_{c,d}) - T(Y_{a,d}) \cdot T(Y_{c,b})$ is the same as the A -module of quadratic relations in the X 's, and the proof will be complete. Call the first module N_1 , and the second N_2 . It is easy to check that $N_1 \subset N_2$, simply by calculating out the expressions $T(Y_{a,b}) \cdot T(Y_{c,d}) - T(Y_{a,d}) \cdot T(Y_{c,b})$.

Consider the diagram

$$\begin{array}{ccccc} N_1 & \xhookrightarrow{i} & N_2 & \hookrightarrow & S^2 V_{2\delta} \otimes A \\ \downarrow & & \downarrow & & \downarrow \\ N_1 \otimes k & \xrightarrow{j} & N_2 \otimes k & \xrightarrow{l} & S^2 V_{2\delta} \otimes k. \end{array}$$

Since N_2 is a direct summand of $S^2 V_{2\delta} \otimes A$ according to the first part of this proof, it follows that l is injective. But in Theorem 2, § 4, we showed that $N_1 = N_2$ in case $A = k$. Therefore the images of $N_1 \otimes k$ and $N_2 \otimes k$ in $S^2 V_{2\delta} \otimes k$ are equal. This implies that j is surjective, hence that i is surjective, and hence $N_1 = N_2$. *Q.E.D.*

§ 7. The 2-Adic Limit

Up to this point, we have been studying pairs (X, L) , consisting of abelian varieties X , and ample invertible sheaves L of separable type. To push the theory further, however, it seems almost essential to make it freer of variations within one isogeny type. The simplest way to do this is to study simultaneously a whole tower of abelian varieties. All the simplifications that occur however have to do with dividing by 2, so it doesn't seem necessary or fruitful to look at *all* isogenies: instead we look only at isogenies of degree 2^n , some n . This is economical, too, because our moduli will then be the values of functions on 2-adic vector spaces, rather than of functions on adelic spaces. As always, we assume $\text{char}(k) \neq 2$.

The basic definition is:

Definition 1. A 2-tower of abelian varieties (or tower) is an inverse system $\{X_\alpha\}_{\alpha \in S}$ of abelian varieties, i.e., S is a partially ordered set such that $\forall \alpha_1, \alpha_2 \in S, \exists \beta \in S, \beta > \alpha_1, \alpha_2$, and whenever $\alpha, \beta \in S, \alpha > \beta$, we are

given an isogeny of degree 2^n :

$$p_{\alpha, \beta}: X_{\alpha} \rightarrow X_{\beta}$$

such that:

- 1) If $\alpha > \beta > \gamma$, the diagram of isogenies

$$\begin{array}{ccc} X_{\alpha} & \xrightarrow{p_{\alpha, \gamma}} & X_{\gamma} \\ & \searrow p_{\alpha, \beta} \quad \nearrow p_{\beta, \gamma} & \\ & X_{\beta} & \end{array}$$

commutes,

- 2) If $\alpha > \beta_1, \beta_2$ and $K_i = \text{kernel } \{X_{\alpha} \rightarrow X_{\beta_i}\}$, then $K_1 \subseteq K_2 \Leftrightarrow \beta_1 \geq \beta_2$.

- 3a) For all $\alpha \in S$, and all isogenies $X_{\alpha} \rightarrow Y$, of degree 2^n (some n), $\exists \beta \in S$ such that $\alpha \geq \beta$, and one has a diagram:

$$\begin{array}{ccc} & Y & \\ & \nearrow & \\ X_{\alpha} & & \\ & \searrow p_{\alpha, \beta} & \\ & X_{\beta} & \end{array}$$

- 3b) For all $\alpha \in S$, and all isogenies $Y \rightarrow X_{\alpha}$ of degree 2^n (some n), $\exists \beta \in S$ such that $\beta \geq \alpha$ and one has a diagram:

$$\begin{array}{ccc} Y & & \\ \parallel & \searrow & \\ X_{\beta} & & X_{\alpha} \\ & \nearrow p_{\beta, \alpha} & \end{array}$$

Note that, starting with *one* abelian variety X , we can generate in a canonical way a 2-tower of abelian varieties by taking coverings and quotients of degrees 2^n , starting with X .

Definition 2. If X is an abelian variety, let $\text{tor}_2(X)$ denote the group of closed points $x \in X$ of order 2^n , some n .

If $\underline{X} = \{X_{\alpha}\}$ is a tower of abelian varieties, we get a derived inverse system of discrete groups $\{\text{tor}_2(X_{\alpha})\}$. Let

$$V(\underline{X}) = \varprojlim_{\alpha \in S} \text{tor}_2(X_{\alpha}).$$

If X is one abelian variety, we also let $V(X)$ denote $V(\underline{X})$, where \underline{X} is the 2-tower generated by X . This is the usual 2-Tate group of X . For all $\alpha \in S$, there is a canonical surjection

$$p_{\alpha}: V(\underline{X}) \rightarrow \text{tor}_2(X_{\alpha}).$$

Denote the kernel by $T(\alpha)$. Each $T(\alpha)$ is an inverse limit of finite groups, so if we topologize $V(\underline{X})$ by taking the $T(\alpha)$'s as a basis of open neighborhoods of the origin, then $V(\underline{X})$ becomes a locally compact group, and each $T(\alpha)$ becomes a compact, open subgroup. From well-known structure theorems, we know that there are topological isomorphisms:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(\alpha) & \longrightarrow & V(\underline{X}) & \longrightarrow & \text{tor}_2(X_\alpha) \longrightarrow 0 \\ & & \wr \parallel & & \wr \parallel & & \wr \parallel \\ 0 & \longrightarrow & (\mathbb{Z}_2)^{2g} & \longrightarrow & (\mathbb{Q}_2)^{2g} & \longrightarrow & \left\{ \begin{array}{l} \text{subgroup of } (\mathbb{Q}/\mathbb{Z})^{2g} \\ \text{of order } 2^n \\ \text{(some } n) \end{array} \right\} \longrightarrow 0 \end{array}$$

(where $g = \dim X$). In particular,

$$S \cong \left\{ \begin{array}{l} \text{partially ordered set of compact,} \\ \text{open subgroups } T \subset (\mathbb{Q}_2)^{2g} \end{array} \right\}.$$

When \underline{X} is generated from one given X , the kernel of the canonical homomorphism $V(\underline{X}) \rightarrow \text{tor}_2(X)$ is usually denoted $T(X)$, and $T(X)$ is also called the 2-Tate group of X .

A final point: when $k = \mathbb{C}$, one can associate to a tower of abelian varieties a *common universal covering space*. In fact, for all $\alpha \geq \beta$, $p_{\alpha\beta}: X_\alpha \rightarrow X_\beta$ induces an isomorphism

$$\hat{p}_{\alpha\beta}: \hat{X}_\alpha \xrightarrow{\sim} \hat{X}_\beta$$

between the universal covering spaces. Therefore let $\hat{\underline{X}}$ stand for a complex vector space canonically isomorphic to each \hat{X}_α . In the classical theory, $\hat{\underline{X}}$ plays a role analogous to $V(\underline{X})$. For all α , there is a canonical surjection $q_\alpha: \hat{\underline{X}} \rightarrow X_\alpha$. Let $L(\alpha)$ denote its kernel. Then $\{L(\alpha)\}$ is a family of commensurable lattices in $\hat{\underline{X}}$, and $\alpha \geq \beta$ if and only if $L(\alpha) \subseteq L(\beta)$. Since $X_\alpha \cong \hat{\underline{X}}/L(\alpha)$, the whole tower can be generated by starting with the complex vector space $\hat{\underline{X}}$ and dividing by these lattices $L(\alpha)$.

Definition 3. A *polarized tower of abelian varieties* is a tower $\{X_\alpha\}$, $\alpha \in S$, plus a set of totally symmetric ample invertible sheaves of degree 2^n , some n : L_α on X_α , for α in a subset $S_0 \subset S$, plus isomorphisms

$$p_{\alpha,\beta}^*(L_\beta) \xrightarrow{\sim} L_\alpha$$

whenever $\alpha, \beta \in S_0$, $\alpha > \beta$. We require:

- 1) If $\alpha, \beta \in S$, and $\alpha > \beta$, then $\beta \in S_0 \Rightarrow \alpha \in S_0$.
- 2) If $\alpha > \beta > \gamma$, $\gamma \in S_0$, then the isomorphisms of $L_\alpha, L_\beta, L_\gamma$ — all pulled up to X_α — are to be compatible,

3) If $\alpha > \beta$, $\alpha \in S_0$, and if there exists a totally symmetric sheaf M on X_β such that $p_{\alpha, \beta}^* M \cong L_\alpha$, then $\beta \in S_0$ too, [in which case, L_β will have to be M too].

The object of this section is to generalize the theory of §1 to a polarized tower of abelian varieties. So from now on, let's suppose given one such tower $\mathcal{T} = \{X_\alpha, L_\alpha\}$. The first important observation is:

"4 > 2" Lemma. For all $x \in V(X)$, there exists an $\alpha \in S_0$ such that $p_\alpha(x) \in H(L_\alpha)$.

Proof. Start with any $\alpha_1 \in S_0$. Let 2^n be the order of the point $p_{\alpha_1}(x)$. Let $\alpha_2 \in S_0$ be the element such that $\alpha_2 > \alpha_1$ and such that the isogeny p_{α_2, α_1} is $2^n \delta$, i.e.,

$$\begin{array}{ccc} X_{\alpha_2} & & \\ \downarrow q & \searrow p_{\alpha_2, \alpha_1} & \\ & X_{\alpha_1} & \\ \uparrow 2^n \delta & & \\ X_{\alpha_1} & & \end{array}$$

Since $(2^n \delta)^* L_{\alpha_1} \cong (L_{\alpha_1})^{2^{2n}}$, $H((2^n \delta)^* L_{\alpha_1})$ contains all points of X_{α_1} of order 2^{2n} . Therefore $H(L_{\alpha_2})$ contains all points of X_{α_2} of order 2^{2n} . But

$$0 = 2^n p_{\alpha_1}(x) = 2^n p_{\alpha_2, \alpha_1}(p_{\alpha_2}(x)) = 2^{2n} [q(p_{\alpha_2}(x))]$$

so $p_{\alpha_2}(x)$ has order 2^{2n} . *Q.E.D.*

Now let $S_0^x = \{\alpha \in S_0 \mid p_\alpha(x) \in H(L_\alpha)\}$. Like S_0 , it has the property: $\beta > \alpha$, $\alpha \in S_0^x \Rightarrow \beta \in S_0^x$. Suppose that for one $\alpha \in S_0^x$, you choose an isomorphism

$$\varphi_\alpha: L_\alpha \xrightarrow{\sim} T_{p_\alpha(x)}^*(L_\alpha).$$

Then I claim this determines canonically isomorphisms

$$\varphi_\beta: L_\beta \xrightarrow{\sim} T_{p_\beta(x)}^*(L_\beta)$$

for all $\beta \in S_0^x$. This is clear — first suppose $\beta > \alpha$, and then choose φ_β so that:

$$\begin{array}{ccc} L_\beta & \xrightarrow{\varphi_\beta} & T_{p_\beta(x)}^*(L_\beta) \\ \parallel & & \parallel \\ & & T_{p_\beta(x)}^*(p_{\beta, \alpha}^*(L_\alpha)) \\ \parallel & & \parallel \\ p_{\beta, \alpha}^*(L_\alpha) & \xrightarrow{p_{\beta, \alpha}^*(\varphi_\alpha)} & p_{\beta, \alpha}^*(T_{p_\alpha(x)}^*(L_\alpha)) \end{array}$$

commutes. In general, first determine φ_γ for some $\gamma \in S_0^x$ such that $\gamma > \alpha, \beta$, then go backwards to determine φ_β from φ_γ . Call such a system of φ_α 's, all $\alpha \in S_0^x$, a *compatible* set of isomorphisms $L_\alpha \xrightarrow{\sim} T_{p_\alpha(x)}^*(L_\alpha)$.

Definition 4. $\mathcal{G}(\mathcal{T}) = \text{set of pairs } (x, \{\varphi_\alpha\})$, where $x \in V(\underline{X})$, and $\{\varphi_\alpha\}, \alpha \in S_0^x$, is a compatible set of isomorphisms.

This forms a group in the usual way. Given $(x, \{\varphi_\alpha\}), (y, \{\psi_\alpha\}) \in \mathcal{G}(\mathcal{T})$, choose any $\gamma \in S_0^x \cap S_0^y$. Then form the composition:

$$L_\gamma \xrightarrow{\psi_\gamma} T_{p_\gamma(y)}^*(L_\gamma) \xrightarrow{T_{p_\gamma(y)}^*(\varphi_\gamma)} T_{p_\gamma(x+y)}^*(L_\gamma).$$

Call this ρ_γ and generate with it a compatible set of isomorphisms $\{\rho_\alpha\}$, all $\alpha \in S_0^{x+y}$. Then let

$$(x, \{\varphi_\alpha\}) \circ (y, \{\psi_\alpha\}) = (x+y, \{\rho_\alpha\}).$$

Moreover, we get an exact sequence:

$$0 \longrightarrow k^* \longrightarrow \mathcal{G}(\mathcal{T}) \xrightarrow{\pi} V(\underline{X}) \longrightarrow 0$$

as usual. Notice the simplification here over the theory of §1: the finite group $H(L)$ that depended on L has been replaced by the big group $V(\underline{X})$ depending only on the tower and not on the polarization. It is not hard to interpret $\mathcal{G}(\mathcal{T})$ as a simultaneous direct and inverse limit of the $\mathcal{G}(L_\alpha)$'s, with respect to the connections induced between them whenever $\alpha > \beta$, as in Prop. 2, §1. In particular, for all $\alpha_1 \in S_0$, if we look at the subgroup:

$$\mathcal{G}_{\alpha_1}^*(\mathcal{T}) = \pi^{-1}[p_{\alpha_1}^{-1}(H(L_{\alpha_1}))]$$

$$\bigcap$$

$$\mathcal{G}(\mathcal{T})$$

then for all $(x, \{\varphi_\alpha\}) \in \mathcal{G}_{\alpha_1}^*(\mathcal{T})$, $p_{\alpha_1}(x) \in H(L_{\alpha_1})$ so φ_{α_1} is defined, and

$$(x, \{\varphi_\alpha\}) \mapsto (p_{\alpha_1}(x), \varphi_{\alpha_1})$$

defines a surjection:

$$\mathcal{G}_{\alpha_1}^*(\mathcal{T}) \xrightarrow{q_{\alpha_1}} \mathcal{G}(L_{\alpha_1}) \longrightarrow 0.$$

Let $K(\alpha_1)$ denote the kernel of this map. We get the picture:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & K(\alpha) & \xrightarrow{\sim} & T(\alpha) & & \\ & & \downarrow & & \downarrow & & \\ \longrightarrow & k^* & \longrightarrow & \mathcal{G}_\alpha^*(\mathcal{T}) & \xrightarrow{\pi} & p_\alpha^{-1}(H(L_\alpha)) & \longrightarrow 0 \\ & \parallel & & \downarrow q_\alpha & & \downarrow p_\alpha & \\ 0 \longrightarrow & k^* & \longrightarrow & \mathcal{G}(L_\alpha) & \xrightarrow{\pi} & H(L_\alpha) & \longrightarrow 0 \\ & & & \downarrow & & \downarrow & \\ & & & 0 & & 0 & \end{array}$$

Lemma 2. For all $\alpha \in S_0$, $\mathcal{G}_\alpha^*(\mathcal{T})$ is the centralizer of $K(\alpha)$ in $\mathcal{G}(\mathcal{T})$, and $k^* \cdot K(\alpha)$ is the center of $\mathcal{G}_\alpha^*(\mathcal{T})$.

Proof. If $x \in K(\alpha)$ and $y \in \mathcal{G}_\alpha^*(\mathcal{T})$, then $y \cdot x \cdot y^{-1}$ is still in $K(\alpha)$, and it has the same image in $T(\alpha)$ as x has, since $p_\alpha^{-1}(H(L_\alpha))$ is a commutative group. Therefore $x = y \cdot x \cdot y^{-1}$, i.e., $K(\alpha) \subset \text{center}[\mathcal{G}_\alpha^*(\mathcal{T})]$. Since k^* is even in the center of $\mathcal{G}(\mathcal{T})$, $k^* \cdot K(\alpha) \subset \text{center}[\mathcal{G}_\alpha^*(\mathcal{T})]$ too. But if $x \in \text{center}[\mathcal{G}_\alpha^*(\mathcal{T})]$, $q_\alpha(x) \in \text{center}[\mathcal{G}(L_\alpha)]$ and we know k^* is the whole center of $\mathcal{G}(L_\alpha)$. Therefore $k^* \cdot K(\alpha)$ is exactly the center of $\mathcal{G}_\alpha^*(\mathcal{T})$. Now suppose $y \in \mathcal{G}(\mathcal{T})$ centralized $K(\alpha)$. y is certainly in $\mathcal{G}_\beta^*(\mathcal{T})$ for some $\beta \in S$, $(\beta > \alpha)$. Then the image $q_\beta(y)$ in $\mathcal{G}(L_\beta)$ commutes with the image $q_\beta[K(\alpha)]$ in $\mathcal{G}(L_\beta)$. Now $q_\beta[K(\alpha)]$ is a subgroup of $\mathcal{G}(L_\beta)$ lying over the subgroup $\text{Ker}(p_{\beta\alpha})$ of $H(L_\beta)$. Therefore by Prop. 2, §1, $q_\beta(y)$ is an element of $\mathcal{G}(L_\beta)$ whose image in $H(L_\beta)$ is in $p_{\beta,\alpha}^{-1}[H(L_\alpha)]$, i.e., the image of y in $V(\underline{X})$ is in $p_{\alpha,\beta}^{-1}(H(L_\alpha))$, or $y \in \mathcal{G}_\alpha^*(\mathcal{T})$. *Q.E.D.*

Corollary. $k^* = \text{center of } \mathcal{G}(\mathcal{T})$.

As in §1, we can describe the non-commutativity by a skew-symmetric form $e_\lambda: V(\underline{X}) \times V(\underline{X}) \rightarrow k^*$:

$$e_\lambda(\pi x, \pi y) = x \cdot y \cdot x^{-1} \cdot y^{-1}$$

all $x, y \in \mathcal{G}(\mathcal{T})$. Then the lemma tells us that:

1) $p_\alpha^{-1}(H(L_\alpha))$ is the group of elements $x \in V(\underline{X})$ such that $e_\lambda(x, y) = 1$, all $y \in T(\alpha)$, i.e., $T(\alpha)^\perp$.

2) $T(\alpha)$ is the group of elements $x \in V(\underline{X})$ such that $e_\lambda(x, y) = 1$, all $y \in p_\alpha^{-1}(H(L_\alpha))$, i.e., the degenerate subspace for the pairing e_λ on $p_\alpha^{-1}(H(L_\alpha))$.

3) For all $x \in V(\underline{X})$, there is a $y \in V(\underline{X})$ such that $e_\lambda(x, y) \neq 1$, i.e., e_λ is non-degenerate.

In particular, for all $\alpha \in S_0$, (a) $e_\lambda \equiv 1$ on $T(\alpha)$, (b) $T(\alpha)^\perp/T(\alpha) \cong H(L_\alpha)$, and (c) the pairing induced on $T(\alpha)^\perp/T(\alpha)$ by e_λ corresponds to the old pairing e_{L_α} on $H(L_\alpha)$.

Using the symmetry of the L_α 's, we obtain an automorphism

$$\delta_{-1}: \mathcal{G}(\mathcal{T}) \rightarrow \mathcal{G}(\mathcal{T})$$

exactly as in §2. But now since $2 \cdot V(\underline{X}) = V(\underline{X})$, δ_{-1} is much more convenient than before. In fact, it induces a *canonical section* of $\mathcal{G}(\mathcal{T})$ over $V(\underline{X})$:

Definition 5. Let $x \in V(\underline{X})$. Let $z \in \mathcal{G}(\mathcal{T})$ satisfy $\pi(z) = x/2$, $\delta_{-1}(z) = z^{-1}$: there are exactly 2 such elements in $\mathcal{G}(\mathcal{T})$, z and $(-1) \cdot z$ (here -1 is an

element of k^* and we multiply in $\mathcal{G}(\mathcal{T})$. Let

$$\sigma(x) = z^2$$

(which is independent of the choice of z).

Therefore, $\mathcal{G}(\mathcal{T})$ decomposes as a set:

$$\mathcal{G}(\mathcal{T}) \cong k^* \times V(\underline{X})$$

if

$$\alpha \cdot \sigma(x) \leftrightarrow (\alpha, x).$$

Let's compute what happens to the group law:

Lemma 3. For all $x, y \in V(\underline{X})$, $\sigma(x) \cdot \sigma(y) = e_\lambda(x, y/2) \cdot \sigma(x+y)$.

Proof. Let $z, w \in \mathcal{G}(\mathcal{T})$ lie over $x/2, y/2$ respectively and satisfy $\delta_{-1} z = z^{-1}$, $\delta_{-1} w = w^{-1}$. Let

$$s = e_\lambda(-x/4, y/2) \cdot z \cdot w.$$

Then s lies over $(x+y)/2$ and satisfies:

$$\begin{aligned} \delta_{-1} s &= e_\lambda(-x/4, y/2) \cdot \delta_{-1} z \cdot \delta_{-1} w \\ &= e_\lambda(-x/4, y/2) \cdot [z^{-1} \cdot w^{-1} \cdot z \cdot w] \cdot (z \cdot w)^{-1} \\ &= e_\lambda(-x/4, y/2) \cdot e_\lambda(-x/2, -y/2) \cdot (z \cdot w)^{-1} \\ &= s^{-1}. \end{aligned}$$

Therefore

$$\begin{aligned} e_\lambda(x, y/2) \sigma(x+y) &= e_\lambda(x, y/2) \cdot s^2 \\ &= e_\lambda(x/2, y/2) \cdot z \cdot w \cdot z \cdot w \\ &= e_\lambda(x/2, y/2) \cdot z^2 (z^{-1} \cdot w \cdot z \cdot w^{-1}) w^2 \\ &= \sigma(x) \cdot \sigma(y). \end{aligned}$$

Q.E.D.

In other words, the group law, carried over to $k^* \times V(\underline{X})$, is

$$(\alpha, x) \cdot (\beta, y) = (\alpha \cdot \beta \cdot e_\lambda(x, y/2), x+y).$$

Let's give a complete structure theorem for $\mathcal{G}(\mathcal{T})$ and $V(\underline{X})$. Let

$\chi: \mathcal{Q}_2 \rightarrow k^*$ be an additive character with kernel \mathbf{Z}_2 .

Definition 6. A symplectic isomorphism of $V(\underline{X})$ and $\mathcal{Q}_2^g \times \mathcal{Q}_2^g$ is an isomorphism \bar{c} such that if $\varphi_1 x, \varphi_2 x$ are the 1^{st} g and 2^{nd} g components of $\bar{c}(x)$, then

$$e_\lambda(x, y) = \chi[\varphi_1 x \cdot \varphi_2 y - \varphi_1 y \cdot \varphi_2 x]$$

for all $x, y \in V(\underline{X})$ (here t denotes the transpose vector, and \cdot is multiplication of $1 \times g$ and $g \times 1$ matrices).

We leave it to the reader to check that such an isomorphism always exists: for example by constructing it inductively via some cofinal series $\alpha_1 < \alpha_2 < \dots$ in S .

Definition 7. $\mathcal{G}_g = k^* \times \mathcal{Q}_2^g \times \mathcal{Q}_2^g$, with group law

$$(\alpha, x, y) \cdot (\beta, u, v) = (\alpha \cdot \beta \cdot \chi({}^t x \cdot v), x + u, y + v).$$

$D_{-1}: \mathcal{G}_g \rightarrow \mathcal{G}_g$ is the automorphism $D_{-1}((\alpha, x, y)) = (\alpha, -x, -y)$.
 $\Sigma: \mathcal{Q}_2^g \times \mathcal{Q}_2^g \rightarrow \mathcal{G}_g$ is the section

$$\Sigma((x, y)) = \left(\chi \left(\frac{{}^t x \cdot y}{2} \right), x, y \right).$$

Definition 8. A full \mathfrak{g} -structure for \mathcal{T} is an isomorphism $c: \mathcal{G}(\mathcal{T}) \xrightarrow{\sim} \mathcal{G}_g$, which is the identity on the subgroups k^* and such that $c \circ \delta_{-1} = D_{-1} \circ c$.

It follows immediately that a full \mathfrak{g} -structure c induces a symplectic isomorphism $\bar{c}: V(\underline{X}) \xrightarrow{\sim} \mathcal{Q}_2^{2g}$, and also that $\Sigma \circ \bar{c} = c \circ \sigma$. Therefore c is even determined by \bar{c} . This is a simplification which was foreshadowed in the discussion at the end of § 2. In fact, given a symplectic isomorphism \bar{c} , we can define the unique full \mathfrak{g} -structure c extending \bar{c} by:

$$\begin{aligned} c(\lambda \cdot \sigma(x)) &= \lambda \cdot \Sigma(\bar{c}(x)) \\ &= \left(\lambda \cdot \chi \left(\frac{{}^t \varphi_1 x \cdot \varphi_2 x}{2} \right), \varphi_1 x, \varphi_2 x \right) \end{aligned}$$

for all $\lambda \in k^*, x \in V(\underline{X})$, and one checks all the requirements easily. In particular, one full \mathfrak{g} -structure always exists and we have a structure theorem for $\mathcal{G}(\mathcal{T})$ and its maps.

So far, in our polarized tower, we have considered only the *totally symmetric* L_α 's that can be put in an inverse system on the X_α 's. For a few more α 's, however, we may be able to find a *symmetric* invertible sheaf L_α on X_α , such that for some element $\beta \in S_0$ for which $\beta \geq \alpha$, $p_{\alpha, \beta}^* L_\alpha$ is isomorphic to the L_β that we already have: such an L_α will be said to be *compatible* with the given polarization.

If we define as before

$$\mathcal{G}_\alpha^*(\mathcal{T}) = \pi^{-1}(p_\alpha^{-1}(H(L_\alpha))),$$

then every element $(\{x_\beta\}, \{\varphi_\beta\}) \in \mathcal{G}_\alpha^*(\mathcal{T})$ is compatible with a unique isomorphism

$$\varphi_\alpha: L_\alpha \xrightarrow{\sim} T_{x_\alpha}^*(L_\alpha)$$

and we obtain a homomorphism $q_\alpha: \mathcal{G}_\alpha^*(\mathcal{T}) \rightarrow \mathcal{G}(L_\alpha)$ just as before. Let $K(\alpha)$ be its kernel: a subgroup of $\mathcal{G}(\mathcal{T})$ isomorphic under π with $T(\alpha)$.

Lemma 4. *If L_α is a symmetric sheaf on X_α compatible with our polarization as above, then $K(\alpha)$ is the set of elements*

$$e_*^{L_\alpha}(p_\alpha(\tfrac{1}{2}x)) \cdot \sigma(x)$$

$x \in T(\alpha)$. In particular, if L_α is totally symmetric, then $K(\alpha) = \sigma[T(\alpha)]$.

Proof. To unwind the definition of $\sigma(x)$, let $\beta > \alpha$ be the element of S giving us a diagram:

$$\begin{array}{ccc} X_\beta & & \\ \parallel & \searrow p_{\beta, \alpha} & \\ X_\alpha & \nearrow 2\delta & X_\alpha \end{array}$$

Then $p_{\beta, \alpha}^* L_\alpha$ is totally symmetric, hence isomorphic to L_β , and $u = p_\beta(\frac{1}{2}x)$ is in $H(L_\beta)$. Choose an isomorphism

$$\psi: L_\beta \xrightarrow{\sim} T_u^* L_\beta$$

such that $\delta_{-1}((u, \psi)) = (u, \psi)^{-1}$. Let $(u, \psi)^2 = (2u, \varphi_\beta)$. Then $\sigma(x)$ is represented by the elements $(2u, \varphi_\alpha) \in \mathcal{G}(L_\alpha)$. Since $x \in T(\alpha)$, $2u \in \text{Ker}(p_{\beta, \alpha})$ and there is a scalar λ such that:

$$\begin{array}{ccc} L_\beta & \xrightarrow{\varphi_\beta} & T_{2u}^* L_\beta \\ \parallel & & \parallel \\ p_{\beta, \alpha}^* L_\alpha & & T_{2u}^* p_{\beta, \alpha}^* L_\alpha \\ & \searrow \text{mult. by } \lambda & \parallel \\ & & p_{\beta, \alpha}^* L_\alpha \end{array}$$

commutes. This means that $\lambda^{-1} \cdot \sigma(x)$ is compatible with the identity map from L_α to L_α , i.e., is in the kernel of q_α . The lemma boils down then to checking that $\lambda = e_*^{L_\alpha}(p_{\beta, \alpha}(u))$. Throwing out irrelevant notation, we can restate this fact as:

Lemma 5. *Let L be a symmetric invertible sheaf of separable type on an abelian variety X . Let $u \in X_4$ and let $(u, \varphi) \in \mathcal{G}((2\delta)^* L)$ satisfy $\delta_{-1}((u, \varphi)) = (u, \varphi)^{-1}$. Let $(u, \varphi)^2 = (2u, \psi)$. Then the composite isomorphism:*

$$(2\delta)^* L \xrightarrow{\psi} T_{2u}^* (2\delta)^* L \cong (2\delta)^* T_{4u}^* L = (2\delta)^* L$$

is multiplication by $e_*^L(2u)$.

Proof. Since L is symmetric, $(2\delta)^* L \cong L^4$. Let $v = 2u$ and let (v, ρ) be the element $\eta_2((u, \varphi))$ in $\mathcal{G}(L^2)$. Since $\eta_2 \circ \delta_{-1} = \delta_{-1} \circ \eta_2$, $\delta_{-1}((v, \rho)) = (v, \rho)^{-1}$. But $e_*^{L^2}(v) = [e_*^L(v)]^2 = 1$, so by Prop. 3, § 2, $\delta_{-1}((v, \rho)) = (v, \rho)$.

Therefore $(v, \rho)^2 = 1$ in $\mathcal{G}(L^2)$. Therefore by Prop. 6, § 2, $\eta_2((v, \rho)) = (0, e_*^L(v))$ in $\mathcal{G}(L)$. Explicitly, this means that:

$$\begin{array}{ccc}
 L^4 & \xrightarrow{\rho^2} & T_v^* L^4 \\
 \parallel & & \parallel \\
 (2\delta)^* L & \dashrightarrow & T_v^* (2\delta)^* L \\
 \text{mult. by } e_*^L(v) \searrow & & \parallel \\
 & & (2\delta)^* T_{2v}^* L \\
 & & \parallel \\
 & & (2\delta)^* L
 \end{array}$$

commutes. But (v, ρ^2) , as an element of $\mathcal{G}(L^4)$, is $\varepsilon_2((v, \rho))$. And

$$\varepsilon_2((v, \rho)) = \varepsilon_2(\eta_2((u, \varphi))) = \delta_2((u, \varphi)) = (u, \varphi)^3 \cdot \delta_{-1}((u, \varphi)) = (v, \psi).$$

Therefore the dotted arrow is ψ and the lemma is proven. *Q.E.D.*

Conversely, suppose we start with any $\alpha \in S$ such that $e \equiv 1$ on $T(\alpha) \times T(\alpha)$, i.e., $T(\alpha)$ is isotropic, and try to make a subgroup of $\mathcal{G}(\mathcal{T})$ via

$$K(\alpha) = \{e_*(\tfrac{1}{2}x) \cdot \sigma(x) \mid x \in T(\alpha)\}$$

where e_* is a function from $\tfrac{1}{2}T(\alpha)$ to $\{\pm 1\}$. This works if we take for e_* any function satisfying

$$e_*(x+y) = e_\lambda(x, y)^2 \cdot e_*(x) \cdot e_*(y).$$

In particular, $e_*(x) = 1$ if $x \in T(\alpha)$. Let $\beta \in S$ be such that $2T(\alpha) = T(\beta)$. Then $\beta \in S_0$, and $K(\beta) = \sigma[T(\beta)] \subset K(\alpha)$. Let L_β be the totally symmetric sheaf on X_β defined by our polarized tower. Then $K(\alpha)/K(\beta)$ is a level subgroup of $\mathcal{G}(L_\beta)$ lying over the subgroup $p_\beta[T(\alpha)]$ of $H(L_\beta)$. As in § 1, it provides descent data for L_β in the isogeny $p_{\beta\alpha}: X_\beta \rightarrow X_\alpha$, since $\text{Ker}(p_{\beta\alpha}) = p_\beta[T(\alpha)]$. Let it define L_α on X_α . This L_α is easily seen to be symmetric, compatible with the polarization, and satisfying

$$e_*^{L_\alpha}(p_\alpha(x)) = e_*(x), \quad \text{all } x \in \tfrac{1}{2}T(\alpha).$$

Conclusion. Symmetric sheaves L_α on some X_α , compatible with the polarization, are in 1-1 correspondence with all possible "level" subgroups $K \subset \mathcal{G}(\mathcal{T})$ such that

- 1) $K \cap k^* = \{1\}$,
- 2) for all $x \in K$, $\sigma(\pi(x)) = \pm x$.

This ends our discussion of the groups that are involved in a polarized tower of abelian varieties. Next, we turn to their representations. The family of vector spaces

$$\{\Gamma(X_\alpha, L_\alpha)\}, \quad \alpha \in S_0$$

forms a direct system, and we define:

$$\Gamma(\mathcal{T}) = \varinjlim_{\alpha \in S_0} \Gamma(X_\alpha, L_\alpha).$$

Just as in §1, $\mathcal{G}(\mathcal{T})$ is represented on the vector space $\Gamma(\mathcal{T})$. Moreover, as in §1, we check that we can recover $\Gamma(X_\alpha, L_\alpha)$ from $\Gamma(\mathcal{T})$ since:

$$\Gamma(X_\alpha, L_\alpha) \cong \left\{ \begin{array}{l} \text{elements of } \Gamma(\mathcal{T}) \\ \text{invariant under } K(\alpha) \end{array} \right\}.$$

[The same holds for any symmetric L_α compatible with the polarization and the corresponding $K(\alpha)$.] What representations does a group like $\mathcal{G}(\mathcal{T})$ have? We make the following restriction:

Definition 9. An *admissible representation* $\alpha \mapsto U_\alpha$ of $\mathcal{G}(\mathcal{T})$ [resp. \mathcal{G}_g] in a vector space V is one in which the subgroup k^* acts via its natural character (i.e., if $\alpha \in k^*$, $U_\alpha = \alpha \cdot (id)_V$), and such that for all $x \in V$:

$$\{\sigma \in \mathcal{G}(\mathcal{T}), \text{ resp. } \mathcal{G}_g \mid U_\sigma x = \text{multiple of } x\},$$

is the inverse image in $\mathcal{G}(\mathcal{T})$ [resp. \mathcal{G}_g] of an *open* subset of $V(X)$ [resp. \mathcal{Q}_2^g].

Prop. 3 of §1 generalizes easily to:

Theorem. \mathcal{G}_g , and hence $\mathcal{G}(\mathcal{T})$, has one and only one irreducible admissible representation. All other admissible representations break up into direct sums of the irreducible one with itself.

The proof of this is roughly as follows: choose a maximal open subgroup $U \subset \mathcal{Q}_2^g$ on which the skew-symmetric form

$$e((x_1; y_1), (x_2; y_2)) = \chi[^t x_1 \cdot y_2 - ^t x_2 \cdot y_1]$$

vanishes identically, such as \mathbb{Z}^{2g} . Then construct eigenvectors in V for the subgroup $k^* \times U$ of \mathcal{G}_g . All other elements in the group permute these eigenvectors and we show that this permutation can be described simply, (independent of V). We leave the details to the reader as they are similar to those in the proof of Prop. 3, §1.

This irreducible representation can be written down like this:

Let

$$\mathcal{H}_g = \left\{ \begin{array}{l} \text{vector space of } k\text{-valued, locally constant} \\ \text{functions } f \text{ on } \mathcal{Q}_2^g, \text{ with compact support} \end{array} \right\}.$$

$$[U_{(\alpha, x, y)} f](z) = \alpha \cdot \chi(^t y \cdot z) \cdot f(x + z)$$

$$\text{if } f \in \mathcal{H}_g, (\alpha, x, y) \in \mathcal{G}_g, z \in \mathcal{Q}_2^g.$$

Notice that the only elements $f \in \mathcal{H}_g$ invariant under the operations $U_{(1,x,y)}$, all $x, y \in \mathbb{Z}_2^g$, are the multiples of the characteristic function:

$$D(z) = \begin{cases} 0, & z \notin \mathbb{Z}_2^g \\ 1, & z \in \mathbb{Z}_2^g. \end{cases}$$

Therefore \mathcal{H}_g is irreducible (i.e., in view of the Theorem, if any subspace canonically attached to the representation is *one*-dimensional, the representation must be irreducible).

On the other hand, we have:

Theorem. $\Gamma(\mathcal{T})$ is an irreducible admissible representation for $\mathcal{G}(\mathcal{T})$.

Proof. It is an admissible representation since every $x \in \Gamma(\mathcal{T})$ is in some $\Gamma(X_\alpha, L_\alpha)$, hence is an eigenvector for $k^* \cdot K(\alpha)$. H is irreducible since if $K(\alpha) \subset \mathcal{G}(\mathcal{T})$ corresponds to a symmetric L_α on X_α of degree 1, then the subspace of $K(\alpha)$ -invariants is isomorphic to $\Gamma(X_\alpha, L_\alpha)$ and this is 1-dimensional. *Q.E.D.*

It follows that if we choose a \mathfrak{g} -structure $c: \mathcal{G}(\mathcal{T}) \rightarrow \mathfrak{g}$, we get a unique isomorphism:

$$\Gamma(\mathcal{T}) \xrightarrow[\sim]{\beta} \mathcal{H}_g$$

such that $\beta(U_z(s)) = U_{c(z)}(\beta(s))$, all $z \in \mathcal{G}(\mathcal{T})$, $s \in \Gamma(\mathcal{T})$. The isomorphisms β extend the isomorphisms β in §1 in the following way:

a. let $\alpha \in S_0$ and let $T(\alpha) \subset V(X)$ be a compact open subgroup such that $\bar{c}(T(\alpha))$ in $\mathbb{Q}_2^g \times \mathbb{Q}_2^g$ is of the form $U \times V$.

b. Then $p_\alpha^{-1}(H(L_\alpha))$ is the orthogonal subgroup, i.e., $V^\perp \times U^\perp$ (if $V^\perp = \{x \in \mathbb{Q}_2^g \mid \chi(x \cdot y) = 1, \text{ all } y \in V\}$, U^\perp similar). Therefore \bar{c} induces isomorphisms:

$$H(L_\alpha) \cong \frac{p_\alpha^{-1}(H(L_\alpha))}{T(\alpha)} \cong (V^\perp/U) \times (U^\perp/V).$$

If we choose an isomorphism

$$(V^\perp/U) \xrightarrow{\sim} K(\delta),$$

then this gives

$$(U^\perp/V) \cong (\widehat{V^\perp/U}) \cong \widehat{K(\delta)},$$

hence

$$H(L_\alpha) \cong K(\delta) \times \widehat{K(\delta)}.$$

c. Thus we get an isomorphism:

$$\begin{aligned}\mathcal{G}(L_\alpha) &\cong \frac{\mathcal{G}_\alpha^*}{K(\alpha)} \\ &\cong k^* \times (V^\perp/U) \times (U^\perp/V) \\ &\cong k^* \times K(\delta) \times \widehat{K(\delta)} = \mathcal{G}(\delta).\end{aligned}$$

d. On the other hand, β restricts as follows:

$$\begin{array}{ccc}\Gamma(\mathcal{T}) & \xrightarrow{\beta} & \mathcal{H}_g \\ \bigcup & & \bigcup \\ \Gamma(\mathcal{T})^{K(\alpha)} & \dashrightarrow & \left\{ \begin{array}{l} \text{functions } \varphi \text{ on } \mathcal{Q}_2^g \\ \text{support in } V^\perp \\ \text{constant w.r.t.} \\ \text{translations by } U \end{array} \right\} \\ \parallel & & \parallel \\ \Gamma(X_\alpha, L_\alpha) & & \{ \text{functions on } V^\perp/U \} \\ & & \parallel \\ & & \{ \text{functions on } K(\delta) \} = V(\delta).\end{array}$$

The induced isomorphism of $\Gamma(X_\alpha, L_\alpha)$ and $V(\delta)$ is exactly the isomorphism β of §1 corresponding to the \mathfrak{g} -structure on $\mathcal{G}(L_\alpha)$ occurring in c.

In short, β is just the union of the isomorphisms β obtained on a finite level previously.

Next choose consistent isomorphisms $(L_\alpha)_0 \otimes \kappa(0) \xrightarrow{\sim} k$, for all $\alpha \in S_0$. These induce “evaluation at 0” maps: $\Gamma(X_\alpha, L_\alpha) \rightarrow k$, for all α , which fit together into one big “evaluation at 0” map:

$$A: \Gamma(\mathcal{T}) \rightarrow k.$$

To describe this piece of information in closed form, we need to describe the dual space \mathcal{H}_g^* :

Dual of \mathcal{H}_g : \mathcal{H}_g^* is spanned by the characteristic functions φ_U of compact, open subsets $U \subset \mathcal{Q}_g^2$, with the obvious relations:

$$(*) \quad \varphi_{U_1 \cup U_2} + \varphi_{U_1 \cap U_2} = \varphi_{U_1} + \varphi_{U_2}.$$

Now let \mathcal{B} be the Boolean algebra of compact open subsets of \mathcal{Q}_2^g . Then a linear functional on \mathcal{H}_g is determined by its values on the φ_U 's, and relations which follow from (*) make this set function into a measure. In other words, if we let

$$\mathcal{H}_g^* = \left\{ \begin{array}{l} \text{vector space of } k\text{-valued finitely additive} \\ \text{measures } \mu \text{ defined on the Boolean algebra } \mathcal{B} \end{array} \right\}$$

then the pairing:

$$\langle f, \mu \rangle = \int_{\mathcal{Q}_2^g} f \cdot d\mu$$

makes \mathcal{H}_g^* into the dual of \mathcal{H}_g .

This shows that there is a unique measure $\mu \in \mathcal{H}_g^*$ such that

$$\Lambda(s) = \int_{\mathcal{Q}_2^g} \beta(s) \cdot d\mu$$

all $s \in \Gamma(\mathcal{T})$. This μ is “built up” out of null functions q_{L_α} in the following way: let $\alpha \in S_0$ be such that $\bar{c}(T(\alpha)) = U \times V$, as above. Then for all $s \in \Gamma(X_\alpha, L_\alpha)$, $\beta(s)$ is a function on \mathcal{Q}_2^g with support in V^\perp , constant on cosets of U . Then

$$\begin{aligned} \Lambda(s) &= \int_{\mathcal{Q}_2^g} \beta(s) \cdot d\mu && \text{via new theory} \\ &= \sum_{x \in V^\perp/U} \beta(s)(x) \cdot \mu(x+U) \end{aligned}$$

while

$$\Lambda(s) = \sum_{x \in K(\delta)} \beta(s)(x) \cdot q_{L_\alpha}(x) \quad \text{via old theory.}$$

Thus identifying $K(\delta)$ with V^\perp/U , we find:

$$q_{L_\alpha}(x \bmod U) = \mu(x+U), \quad \text{all } x \in V^\perp.$$

There are formulae for the other q_{L_α} 's, but they are much more complicated.

Note that since the q_{L_α} 's are known to be even functions, this formula implies that μ is an *even* measure on \mathcal{Q}_2^g .

§ 8. 2-Adic Theta Functions

The basic idea of theta functions is to trivialize an ample sheaf L on an abelian variety X , after pulling it back to some auxiliary space V via a map $\pi: V \rightarrow X$; then sections of L on X become actual k -valued functions on V . In our case, let $\mathcal{T} = \{X_\alpha, L_\alpha\}$ be a polarized 2-tower of abelian varieties, and take $V = V(\underline{X})$. First, what is the “pull-back” of the L_α to V ?

Let $x = \{x_\alpha\} \in V(\underline{X})$. Then for all $\alpha \in S$, let $L_\alpha(x_\alpha) = (L_\alpha)_{x_\alpha} \otimes_{\mathcal{O}_{X_\alpha}} \kappa(x_\alpha)$ as usual. For all $\alpha \geq \beta$, $p_{\alpha\beta}^*$ induces an isomorphism

$$L_\beta(x_\beta) \xrightarrow[p_{\alpha\beta}^*]{\sim} L_\alpha(x_\alpha).$$

Passing to the limit, let $L(x)$ denote the vector space you get which is canonically isomorphic to all $L_\alpha(x_\alpha)$'s. The collection $\{L(x)\}$ of 1-dimen-

sional vector spaces represents “the sheaf induced on $V(\underline{X})$ by the L_α ’s”. In particular, every $s \in \Gamma(\mathcal{T})$ has “values” $s(x) \in L(x)$ for each $x \in V(\underline{X})$.

Now let $z = (x, \{\varphi_\alpha\}) \in \mathcal{G}(\mathcal{T})$. For all α for which φ_α is defined, it induces an isomorphism

$$\varphi_\alpha(x_\alpha): L_\alpha(y) \xrightarrow{\sim} T_{x_\alpha}^*(L_\alpha)(y) = L_\alpha(y + x_\alpha),$$

all $y \in X_\alpha$. In particular, z defines an isomorphism $z_y: L(y) \xrightarrow{\sim} L(x + y)$ for all $y \in V(\underline{X})$. Choose as before an isomorphism λ_0 of $L(0)$ with k .

Define λ_y to be the composite:

$$L(y) \xrightarrow[\sigma(-y)_y]{\sim} L(0) \xrightarrow[\lambda_0]{\sim} k.$$

Then $\{\lambda_y\}$ is a “trivialization” of L_α pulled back to $V(\underline{X})$. In particular, if $s \in \Gamma(\mathcal{T})$, then $\lambda_y(s(y))$ is the value of s at y . Define:

$$\vartheta_{[s]}(y) = \lambda_y(s(y)).$$

This is the *algebraic theta function associated to s* . Alternatively, it can be expressed as:

$$\vartheta_{[s]}(y) = \Lambda[U_{\sigma(-y)}(s)].$$

Property I.

$$\vartheta_{[U_w(s)]}(y) = \alpha \cdot e_\lambda(x/2, y) \cdot \vartheta_{[s]}(y - x), \quad \text{if } w = \alpha \cdot \sigma(x).$$

Proof.

$$\begin{aligned} \vartheta_{[U_w(s)]}(y) &= \Lambda[U_{\sigma(-y)}(U_w(s))] \\ &= \alpha \cdot \Lambda[U_{\sigma(-y) \cdot \sigma(x)}(s)] \\ &= \alpha \cdot e_\lambda(x/2, y) \cdot \Lambda[U_{\sigma(-y+x)}(s)] \\ &= \alpha \cdot e_\lambda(x/2, y) \vartheta_{[s]}(y - x). \end{aligned}$$

Corollary. $\vartheta_{[s]}$ is a locally constant function on $V(\underline{X})$. In fact, if $s \in \Gamma(X_\alpha, L_\alpha)$, and L_α is symmetric associated to $e_*: \frac{1}{2}T(\alpha) \rightarrow \{\pm 1\}$, then $\vartheta_{[s]}(y) = e_*(x/2) e_\lambda(x/2, y) \cdot \vartheta_{[s]}(y - x)$ for all $x \in T(\alpha)$.

Proof. Use $U_w(s) = s$, whenever $w = e_*(x/2) \cdot \sigma(x)$, $x \in T(\alpha)$.

Property II. If $s_1, \dots, s_n \in \Gamma(X_\alpha, L_\alpha)$, and $p(X_1, \dots, X_n) \in k[X_1, \dots, X_n]$ is homogeneous of degree d , then

$$P(s_1, \dots, s_n) = 0 \quad \text{in } \Gamma(X_\alpha, L_\alpha^d)$$

if and only if

$$P(\vartheta_{[s_1]}(y), \dots, \vartheta_{[s_n]}(y)) = 0, \quad \text{all } y \in V(\underline{X}).$$

Equivalently, the map

$$\mathfrak{g}: \Gamma(\mathcal{T}) \rightarrow \left\{ \begin{array}{l} \text{vector space of locally constant} \\ k\text{-valued functions on } V(X) \end{array} \right\}$$

extends to an injective homomorphism:

$$\mathfrak{g}: \bigoplus_{d=0}^{\infty} \varinjlim_{\alpha} \Gamma(X_{\alpha}, L_{\alpha}^d) \rightarrow \left\{ \begin{array}{l} \text{ring of locally constant} \\ k\text{-valued functions on } V(X) \end{array} \right\}.$$

Proof. The fact that \mathfrak{g} extends to a homomorphism, or that $P(s_1, \dots, s_n)=0$ implies $P(\mathfrak{g}_{[s_1]}(y), \dots, \mathfrak{g}_{[s_n]}(y))=0$ follows from the fact that $\mathfrak{g}_{[t]}(y)$ is defined as the *value* of t at y , (when L_{α} is suitably trivialized). The fact that \mathfrak{g} is injective is equivalent to noting that no non-zero section $t \in \Gamma(X_{\alpha}, L_{\alpha}^d)$ can vanish on *all* points $\text{tor}_2(X_{\alpha})$; and this is clear since $\text{tor}_2(X_{\alpha})$ is Zariski-dense in X_{α} . *Q.E.D.*

Property III. If σ is an automorphism of the field k , leaving fixed $k_0 \subset k$, and if (X_{α}, L_{α}) is defined over k_0 , then σ acts on

$$1) \Gamma(X_{\alpha}, L_{\alpha})$$

$$2) V(X)$$

and for all $s \in \Gamma(X_{\alpha}, L_{\alpha})$, $y \in V(X)$

$$\mathfrak{g}_{[\sigma(s)]}(\sigma(y)) = \sigma[\mathfrak{g}_{[s]}(y)].$$

(More generally, it seems reasonable to expect that $\mathfrak{g}_{[s]}$ should “be defined” — in a suitable sense — over any ring R over which an X_{α} and L_{α} are given.)

The proof is straightforward.

Property IV. If $\bar{c}: V(X) \xrightarrow{\sim} \mathcal{Q}_2^g \times \mathcal{Q}_2^g$ is a symplectic isomorphism, the transformation T in the commutative diagram:

$$\begin{array}{ccc} & & \mathcal{H}_g^* \\ & \nearrow \beta & \downarrow T \\ \Gamma(\mathcal{T}) & & \left\{ \begin{array}{l} \text{vector space of all} \\ \text{locally constant} \\ \text{functions on } V(X) \end{array} \right\} \\ & \searrow \mathfrak{g} & \end{array}$$

is given by:

$$T(f)(y) = \int_{\mathcal{Q}_2^g} \chi \left({}^t \varphi_2 y \cdot \left(\frac{\varphi_1 y}{2} - u \right) \right) \cdot f(u - \varphi_1 y) \cdot d\mu_u$$

where $\bar{c}(x) = (\varphi_1 x, \varphi_2 x)$, and $\mu \in \mathcal{H}_g^*$ is the theta-null measure of the last section.

Proof. If $f = \beta(s)$, then

$$\begin{aligned}
 T(f)(x) &= \vartheta_{[s]}(x) \\
 &= A(U_{\sigma(-x)}(s)) \\
 &= \int_{\mathbf{Q}_2^g} (U_{\Sigma(-\varphi_1 x, -\varphi_2 x)} f) \cdot d\mu \\
 &= \int_{\mathbf{Q}_2^g} \chi\left(\frac{{}^t\varphi_1 x \cdot \varphi_2 x}{2}\right) \cdot \chi(-{}^t u \cdot \varphi_2 x) \cdot f(u - \varphi_1 x) \cdot d\mu_u. \quad Q.E.D.
 \end{aligned}$$

Corollary. A basis of $\vartheta[\Gamma(\mathcal{T})]$ is given by the “classical” theta-functions:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} = T \left(\delta \begin{bmatrix} a \\ b \end{bmatrix} \right)$$

where

$$\delta \begin{bmatrix} a \\ b \end{bmatrix} (x) = \begin{cases} 0 & \text{if } x \notin a + \mathbf{Z}_2^g \\ \chi({}^t b \cdot x) & \text{if } x \in a + \mathbf{Z}_2^g \end{cases}$$

and a, b run through coset representatives of $\mathbf{Q}_2^g / \mathbf{Z}_2^g$. Here

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (x) = \chi\left(\left(\frac{{}^t\varphi_2 x}{2} - b\right) \cdot \varphi_1 x\right) \cdot \int_{a + \varphi_1 x + \mathbf{Z}_2^g} \chi({}^t(b - \varphi_2 x) \cdot u) d\mu_u.$$

In particular,

$$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (x) = \chi\left(\frac{{}^t\varphi_2 x \cdot \varphi_1 x}{2}\right) \int_{\varphi_1 x + \mathbf{Z}_2^g} \chi(-{}^t\varphi_2 x \cdot u) d\mu_u$$

is known as RIEMANN’S theta function. It is the transform via T of the characteristic function $\delta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ of \mathbf{Z}_2^g . The multiples of $\delta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ form the subspace of \mathcal{H}_g invariant under the subgroup $\{1\} \times \mathbf{Z}_2^g \times \mathbf{Z}_2^g \subset \mathcal{G}_g$. Therefore, the multiples of $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are exactly the elements of $\vartheta[\Gamma(\mathcal{T})]$ invariant under the subgroup

$$K(\alpha) = c^{-1}(\{1\} \times \mathbf{Z}_2^g \times \mathbf{Z}_2^g) \subset \mathcal{G}(\mathcal{T}).$$

If $T(\alpha)$ is the compact open subgroup of $V(\underline{X})$:

$$T(\alpha) = \bar{c}^{-1}(\mathbf{Z}_2^g \times \mathbf{Z}_2^g),$$

then in fact $K(\alpha) = \{e_*(\frac{1}{2}x) \cdot \sigma(x) \mid x \in T(\alpha)\}$ where

$$e_*(x) = \chi(2{}^t\varphi_1 x \cdot \varphi_2 x), \quad \text{all } x \in \frac{1}{2}T(\alpha).$$

Thus $K(\alpha)$ defines a symmetric invertible sheaf L_α on the abelian variety X_α , and $\Gamma(X_\alpha, L_\alpha)$ is just the space of $K(\alpha)$ -invariants in $\Gamma(\mathcal{T})$. Since this

is 1-dimensional, L_α has degree 1, and $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is nothing but the algebraic theta function defined by the unique section of L_α (up to scalars). Another way to put it is that

$$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \circ \bar{c}^{-1}$$

is the unique function f of the form $\vartheta_{[\bar{c}]} \circ \bar{c}^{-1}$ to satisfy the functional equation:

$$f(x+a) = \chi \left(\frac{{}^t a_1 \cdot a_2}{2} \right) \cdot \chi \left(\frac{{}^t a_1 \cdot x_2}{2} - \frac{{}^t a_2 \cdot x_1}{2} \right) \cdot f(x)$$

all $x \in Q_2^{\mathbb{Z}_2^*}$, $a \in Z_2^{\mathbb{Z}_2^*}$. Note that since μ is an *even* measure on $Q_2^{\mathbb{Z}_2}$, $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an *even* function on $V(X)$:

$$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (-x) = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (x).$$

An important fact is that μ can be reconstructed from $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

Lemma 1. *There is a 1-1 correspondence between*

(A.) *measures $\mu \in \mathcal{H}_g^*$ on $Q_2^{\mathbb{Z}_2}$,*

(B.) *k -valued functions ϑ on $Q_2^{\mathbb{Z}_2} \times Q_2^{\mathbb{Z}_2}$ such that*

$$\vartheta(x+a) = \chi \left(\frac{{}^t a_1 \cdot a_2}{2} \right) \cdot \chi \left(\frac{{}^t a_1 \cdot x_2}{2} - \frac{{}^t a_2 \cdot x_1}{2} \right) \cdot \vartheta(x)$$

all $a \in Z_2^{\mathbb{Z}_2} \times Z_2^{\mathbb{Z}_2}$.

This is set up by:

$$\vartheta(x) = \chi \left(\frac{{}^t x_1 \cdot x_2}{2} \right) \cdot \int_{x_1 + Z_2^{\mathbb{Z}_2}} \chi(-{}^t x_2 \cdot u) \cdot d\mu_u$$

$$\mu(a_1 + 2^n Z_2^{\mathbb{Z}_2}) = 2^{-ng} \cdot \sum_{a_2 \in 2^{-n} Z_2^{\mathbb{Z}_2} / Z_2^{\mathbb{Z}_2}} \chi \left(\frac{{}^t a_1 \cdot a_2}{2} \right) \cdot \vartheta(a_1, a_2).$$

Proof. Left to reader.

In particular, in studying the μ 's that arise from abelian varieties, it is often convenient to go back and forth between μ 's and ϑ 's that correspond as in this lemma. Let's consider RIEMANN's theta relation from this point of view. Before proving it, I want to set it up in both its μ and ϑ -form:

Lemma 2. *Let μ and ϑ be even measures/functions corresponding as in Lemma 1. The following conditions are equivalent:*

(A.) There is a 2^{nd} measure, $\nu \in \mathcal{H}_g^*$ related to μ by the identity:

$$\mu \times \mu(U) = \nu \times \nu(\xi(U))$$

for all compact open subsets $U \subset \mathcal{Q}_2^g \times \mathcal{Q}_2^g$, where ξ is the automorphism of $\mathcal{Q}_2^g \times \mathcal{Q}_2^g$:

$$\xi(x_1, x_2) = (x_1 + x_2, x_1 - x_2).$$

(B.) For all $x, y, u, v \in \mathcal{Q}_2^g \times \mathcal{Q}_2^g$, if $r = -\frac{1}{2}(x + y + u + v)$,

$$\begin{aligned} & \vartheta(x) \cdot \vartheta(y) \cdot \vartheta(u) \cdot \vartheta(v) \\ &= 2^{-g} \cdot \sum_{\eta \in \frac{1}{2} \mathbb{Z}_2^g / \mathbb{Z}_2^g} \chi({}^t r_1 \cdot \eta_2 - {}^t \eta_1 \cdot r_2) \cdot \vartheta(x + r + \eta) \times \\ & \quad \times \vartheta(y + r + \eta) \cdot \vartheta(u + r + \eta) \cdot \vartheta(v + r + \eta). \end{aligned}$$

Proof. To analyze (A.), note first that it is equivalent to the existence of a measure ν such that

$$\begin{aligned} & \int_{\mathcal{Q}_2^g \times \mathcal{Q}_2^g} \xi^* \left(\delta \begin{bmatrix} 2x_1 \\ -x_2 \end{bmatrix} \times \delta \begin{bmatrix} 2y_1 \\ -y_2 \end{bmatrix} \right) d\mu \times d\mu \\ &= \int_{\mathcal{Q}_2^g \times \mathcal{Q}_2^g} \delta \begin{bmatrix} 2x_1 \\ -x_2 \end{bmatrix} \times \delta \begin{bmatrix} 2y_1 \\ -y_2 \end{bmatrix} d\nu \times d\nu \end{aligned}$$

for all $x_1, y_1, x_2, y_2 \in \mathcal{Q}_2^g$. (Here ξ^* denotes pull-back of functions.) This is because the functions

$$\delta \begin{bmatrix} a \\ b \end{bmatrix} \times \delta \begin{bmatrix} c \\ d \end{bmatrix}$$

span the vector space of locally constant function on \mathcal{Q}_2^{2g} with compact support. But

$$\begin{aligned} \xi^* \left(\delta \begin{bmatrix} 2x_1 \\ -x_2 \end{bmatrix} \times \delta \begin{bmatrix} 2y_1 \\ -y_2 \end{bmatrix} \right) (u, v) &= 0 \quad \text{if } u + v \notin 2x_1 + \mathbb{Z}_2^g \\ & \quad \text{or if } u - v \notin 2y_1 + \mathbb{Z}_2^g \\ &= \chi(-{}^t x_2 \cdot (u + v)) \cdot \chi(-{}^t y_2 \cdot (u - v)) \\ & \quad \text{if } u \in x_1 + y_1 + \eta + \mathbb{Z}_2^g \\ & \quad \quad v \in x_1 - y_1 + \eta + \mathbb{Z}_2^g \\ & \quad \text{for some } \eta \in \frac{1}{2} \mathbb{Z}_2^g. \end{aligned}$$

Thus

$$\xi^* \left(\delta \begin{bmatrix} 2x_1 \\ -x_2 \end{bmatrix} \times \delta \begin{bmatrix} 2y_1 \\ -y_2 \end{bmatrix} \right) = \sum_{\eta \in \frac{1}{2} \mathbb{Z}_2^g / \mathbb{Z}_2^g} \delta \begin{bmatrix} x_1 + y_1 + \eta \\ -x_2 - y_2 \end{bmatrix} \times \delta \begin{bmatrix} x_1 - y_1 + \eta \\ -x_2 + y_2 \end{bmatrix}.$$

But note that

$$\begin{aligned} \int_{Q_2^g} \delta \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} d\mu &= \int_{x_1 + Z_2^g} \chi(-{}^t x_2 \cdot u) d\mu_u \\ &= \chi \left(-\frac{{}^t x_1 \cdot x_2}{2} \right) \cdot \vartheta(x_1, x_2). \end{aligned}$$

Given ν , define

$$\Phi(x_1, x_2) = \chi({}^t x_1 \cdot x_2) \int_{2x_1 + Z_2^g} \chi(-{}^t x_2 \cdot u) d\nu_u.$$

Then equation (*) is the same as:

$$\begin{aligned} (**) \quad \sum_{\eta \in \frac{1}{2} Z_2^g / Z_2^g} \chi(-{}^t \eta \cdot x_2) \vartheta(x_1 + y_1 + \eta, x_2 + y_2) \vartheta(x_1 - y_1 + \eta, x_2 - y_2) \\ = \Phi(x_1, x_2) \cdot \Phi(y_1, y_2). \end{aligned}$$

The reader can check that if there is a function Φ satisfying (**), this Φ comes from a ν satisfying (*). Thus (A.) is equivalent to the existence of a Φ satisfying (**). If we let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $\zeta_1 = (\eta, 0)$, then (**) becomes:

$$(**)' \quad \sum_{\zeta_1 \in \frac{1}{2} (Z_2^g \times 0) / Z_2^g \times 0} \chi(-{}^t \zeta_1 \cdot x_2) \vartheta(x + y + \zeta_1) \vartheta(x - y + \zeta_1) = \Phi(x) \cdot \Phi(y).$$

The existence of a Φ satisfying this is clearly equivalent to:

$$\begin{aligned} &\sum_{\zeta_1} \chi(-{}^t \zeta_1 \cdot x_2) \vartheta(x + y + \zeta_1) \vartheta(x - y + \zeta_1) \times \\ &\quad \times \sum_{\zeta_1} \chi(-{}^t \zeta_1 \cdot u_2) \vartheta(u + v + \zeta_1) \vartheta(u - v + \zeta_1) \\ (***) \quad &= \sum_{\zeta_1} \chi(-{}^t \zeta_1 \cdot x_2) \vartheta(x + v + \zeta_1) \vartheta(x - v + \zeta_1) \times \\ &\quad \times \sum_{\zeta_1} \chi(-{}^t \zeta_1 \cdot u_2) \vartheta(u + y + \zeta_1) \vartheta(u - y + \zeta_1) \end{aligned}$$

for all $x, y, u, v \in Q_2^{2g}$ (the summations being as in (**)').

In (***), replace x by $x + \zeta_2$, y by $y + \zeta_2$, u by $u + \zeta_2'$ and v by $v + \zeta_2'$, where $\zeta_2, \zeta_2' \in 0 \times (\frac{1}{2} Z_2^g)$. Multiply by $\chi({}^t \zeta_2 \cdot x_1 + y_1) \cdot \chi({}^t \zeta_2' \cdot u_1 + v_1)$ and sum over all $\zeta_2, \zeta_2' \bmod 0 \times Z_2^g$. Then out comes (B)! Reversing this, you can get (*** out of (B). *Q.E.D.*

Definition. Even measures μ with property (A.) above will be called *Gaussian*.

We now intend to prove that if

$$\vartheta = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and μ is the null-value measure, arising from a polarized tower \mathcal{T} , plus ϑ -structure c , then these conditions on μ and ϑ hold. This fact is derived

by comparing

$$\mathcal{T} = \{X_\alpha, L_\alpha\}$$

with the new tower

$$\mathcal{T}^{(2)} = \{X_\alpha, L_\alpha^2\}.$$

Notice, incidentally, that the relationship between the towers \mathcal{T} and $\mathcal{T}^{(2)}$ is symmetric, in that if we let

$$\mathcal{T}^{(4)} = \{X_\alpha, L_\alpha^4\}$$

then $\mathcal{T}^{(4)}$ is *isomorphic* to \mathcal{T} again. In fact, for all $\alpha \in S$, let $2 * \alpha$ be the new index such that $2 * \alpha > \alpha$ and such that we get a diagram:

$$\begin{array}{ccc} X_{2 * \alpha} & \xrightarrow{\sim} & X_\alpha \\ p_{2 * \alpha, \alpha} \downarrow & \searrow f_\alpha & \\ X_\alpha & & \end{array} \quad \begin{array}{c} \\ \\ 2\delta \end{array}$$

Putting together all these isomorphisms, we find

$$L_{2 * \alpha} \cong f_\alpha^*(L_\alpha^4), \text{ hence } (X_{2 * \alpha}, L_{2 * \alpha}) \cong (X_\alpha, L_\alpha^4)$$

so

$$\mathcal{T} = \{X_{2 * \alpha}, L_{2 * \alpha}\} \cong_f \{X_\alpha, L_\alpha^4\} = \mathcal{T}^{(4)}.$$

The natural homomorphisms ε_2 and η_2 can be defined as in § 2, and we get a diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}(\mathcal{T}) & \longrightarrow & V(X) \longrightarrow 0 \\ & & \alpha^2 \downarrow & & \downarrow \varepsilon_2 & & \parallel \\ 1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}(\mathcal{T}^{(2)}) & \longrightarrow & V(X) \longrightarrow 0 \\ & & \alpha^2 \downarrow & & \downarrow \varepsilon_2 & & \parallel \\ 1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}(\mathcal{T}^{(4)}) & \longrightarrow & V(X) \longrightarrow 0 \\ & & \parallel & & \downarrow \text{via } f & & \downarrow \text{mult. by 2} \\ 1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}(\mathcal{T}) & \longrightarrow & V(X) \longrightarrow 0, \end{array}$$

and everything commutes with δ_{-1} and σ . In terms of our standard groups, this diagram goes over to:

$$\begin{array}{ccccccc} 1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}_g & \longrightarrow & Q_2^{2g} \longrightarrow 0 \\ & & \alpha^2 \downarrow & & \downarrow E_2 & & \parallel \\ 1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}_g^{(2)} & \longrightarrow & Q_2^{2g} \longrightarrow 0 \\ & & \alpha^2 \downarrow & & \downarrow E_2 & & \parallel \\ 1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}_g^{(4)} & \longrightarrow & Q_2^{2g} \longrightarrow 0 \\ & & \parallel & & \downarrow F \wr & & \downarrow 2 \\ 1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}_g & \longrightarrow & Q_2^{2g} \longrightarrow 0 \end{array}$$

where $\mathcal{G}_g^{(2)}$ and $\mathcal{G}_g^{(4)}$ are both equal to $k^* \times \mathcal{Q}_2^g \times \mathcal{Q}_2^g$, but with group laws:

$$\text{resp. } (\alpha, x, y) \cdot (\alpha', x', y') = (\alpha \cdot \alpha' \cdot \chi(2^t x \cdot y'), x + x', y + y')$$

$$= (\alpha \cdot \alpha' \cdot \chi(4^t x \cdot y'), x + x', y + y')$$

and

$$E_2(\alpha, x, y) = (\alpha^2, x, y),$$

$$F(\alpha, x, y) = (\alpha, 2x, 2y).$$

Thus given one symplectic isomorphism

$$V(\underline{X}) \rightarrow \mathcal{Q}_2^g \times \mathcal{Q}_2^g,$$

$$x \mapsto (\varphi_1 x, \varphi_2 x)$$

we get symmetric theta structures

$$c_1: \mathcal{G}(\mathcal{T}) \longrightarrow \mathcal{G}_g, \quad c_2: \mathcal{G}(\mathcal{T}^{(2)}) \xrightarrow{\sim} \mathcal{G}_g^{(2)} \quad \text{and} \quad c_4: \mathcal{G}(\mathcal{T}^{(4)}) \xrightarrow{\sim} \mathcal{G}_g^{(4)}.$$

Via the theta function representation, we obtain injections:

$$\mathfrak{g}: \Gamma(\mathcal{T}) \rightarrow \left\{ \begin{array}{l} \text{functions} \\ \text{on } V(\underline{X}) \end{array} \right\},$$

$$\mathfrak{g}^{(2)}: \Gamma(\mathcal{T}^{(2)}) \rightarrow \left\{ \begin{array}{l} \text{functions} \\ \text{on } V(\underline{X}) \end{array} \right\}$$

and it is easy to check that, for all $s_1, s_2 \in \Gamma(X_\alpha, L_\alpha)$

$$\mathfrak{g}^{(2)}(s_1 \otimes s_2) = \mathfrak{g}(s_1) \cdot \mathfrak{g}(s_2),$$

i.e., tensor product of sections becomes pointwise multiplication of theta functions (compare Property II above).

Now, define actions of both $\mathcal{G}(\mathcal{T})$ and $\mathcal{G}(\mathcal{T}^{(2)})$ on the vector space of all k -valued functions on $V(\underline{X})$ by:

$$(U_w(\varphi))y = \alpha \cdot e_\lambda \left(\frac{x}{2}, y \right) \cdot \varphi(y - x)$$

if

$$w = \alpha \cdot \sigma(x) \in \mathcal{G}(\mathcal{T}),$$

$$(U_w^{(2)}(\varphi))y = \alpha \cdot e_\lambda(x, y) \cdot \varphi(y - x)$$

if

$$w = \alpha \cdot \sigma(x) \in \mathcal{G}(\mathcal{T}^{(2)}).$$

Then according to Property I of algebraic theta functions, $\text{Image}(\mathfrak{g})$ is an irreducible $\mathcal{G}(\mathcal{T})$ -space, and $\text{Image}(\mathfrak{g}^{(2)})$ is an irreducible $\mathcal{G}(\mathcal{T}^{(2)})$ -space. Moreover, $\text{Image}(\mathfrak{g})$ must be generated by the various functions

$$U_w \left(\mathfrak{g} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right),$$

i.e.,

$$y \mapsto e_\lambda \left(\frac{x}{2}, y \right) \cdot \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y-x)$$

and Image $(\vartheta^{(2)})$ must be generated by their various products:

$$y \mapsto e_\lambda \left(\frac{x_1+x_2}{2}, y \right) \cdot \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y-x_1) \cdot \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y-x_2).$$

Now it is a non-trivial condition that this second family of functions spans an *irreducible* $\mathcal{G}(\mathcal{T}^{(2)})$ -space. In particular, let $K \subset \mathcal{G}(\mathcal{T}^{(2)})$ be the subgroup:

$$c_2^{-1}[\{(1, x, y) \mid x \in \frac{1}{2}Z_2^g, y \in Z_2^g\}].$$

Then in an irreducible $\mathcal{G}(\mathcal{T}^{(2)})$ -space, K has a one-dimensional space of invariants. Now it is easy to check that all the functions

$$y \mapsto \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y+x) \cdot \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y-x)$$

(any $x \in V(\underline{X})$) are invariant under $c_2^{-1}(1 \times Z_2^g \times Z_2^g)$. Let $V_1 \subset V(\underline{X})$ be the subgroup $\bar{c}_2^{-1}(Z_2^g \times \{0\})$. Then the functions:

$$y \mapsto \sum_{\zeta \in \frac{1}{2}V_1/V_1} e_\lambda(y, \zeta) \cdot \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y+x+\zeta) \cdot \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y-x+\zeta)$$

are all invariant under K . It follows that they are all proportional to *one* function $\Phi(y)$. Therefore, there are constants, depending on x — call them $c(x)$ — such that

$$\sum_{\zeta \in \frac{1}{2}V_1/V_1} e_\lambda(y, \zeta) \cdot \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y+x+\zeta) \cdot \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y-x+\zeta) = c(x) \cdot \Phi(y).$$

Interchanging x and y in this expression, using the evenness of $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and its periodicity with respect to elements of V_1 , you check that the left-hand side is symmetric in x and y . Thus

$$c(x) \cdot \Phi(y) = c(y) \cdot \Phi(x), \quad \text{all } x, y \in V(\underline{X}).$$

Since neither c nor Φ can be identically 0, this implies that $c(x) = \alpha \cdot \Phi(x)$, for all x and some $\alpha \in k^*$. Replacing Φ by $\sqrt{\alpha} \cdot \Phi$, we get $c = \Phi$, or

$$\sum_{\zeta \in \frac{1}{2}V_1/V_1} e_\lambda(y, \zeta) \cdot \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y+x+\zeta) \cdot \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y-x+\zeta) = \Phi(x) \cdot \Phi(y).$$

This is equation $(**)$ ' in the proof of Lemma 2, so referring to this proof, we see that we have proven that $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ satisfies RIEMANN's theta relation.

Because of the central significance of this result, I want to give a second proof, following the lines of the proof in § 3 in the finite case.

First, introduce the maps:

$$\begin{array}{ccc} \Gamma(\mathcal{T}) & \xrightarrow{\beta} & \mathcal{H}_g \\ & \searrow \vartheta & \downarrow T \\ & & \left\{ \begin{array}{c} \text{functions} \\ \text{on } V(X) \end{array} \right\} \end{array}, \quad \begin{array}{ccc} \Gamma(\mathcal{T}^{(2)}) & \xrightarrow{\beta^{(2)}} & \mathcal{H}_g \\ & \searrow \vartheta^{(2)} & \downarrow T^{(2)} \\ & & \left\{ \begin{array}{c} \text{functions} \\ \text{on } V(X) \end{array} \right\} \end{array}.$$

These induce a pairing

$$\circ: \mathcal{H}_g \times \mathcal{H}_g \rightarrow \mathcal{H}_g$$

such that equivalently

$$\beta(s_1) \circ \beta(s_2) = \beta^{(2)}(s_1 \otimes s_2), \quad s_i \in \Gamma(\mathcal{T})$$

or

$$T(f_1) \cdot T(f_2) = T^{(2)}(f_1 \circ f_2), \quad f_i \in \mathcal{H}_g.$$

Notice that the algebraic tensor product $\mathcal{H}_g \otimes \mathcal{H}_g$ is just \mathcal{H}_{2g} . In particular, the map

$$f_1, f_2 \mapsto f_1 \circ f_2(0)$$

is a linear functional on $\mathcal{H}_g \otimes \mathcal{H}_g$, hence it is represented by a finitely additive measure λ on $\mathcal{Q}_2^g \times \mathcal{Q}_2^g$: i.e.,

$$f_1 \circ f_2(0) = \int_{\mathcal{Q}_2^g \times \mathcal{Q}_2^g} f_1(u) \cdot f_2(v) \cdot d\lambda_{u,v}.$$

Since for all $s_1, s_2 \in \Gamma(\mathcal{T})$, $\alpha \in \mathcal{G}(\mathcal{T})$, $U_\alpha s_1 \otimes U_\alpha s_2 = U_{e_2(\alpha)}(s_1 \otimes s_2)$, we find that for all $f_1, f_2 \in \mathcal{H}_g$, $\alpha \in \mathcal{G}_g$, $U_\alpha f_1 \circ U_\alpha f_2 = U_{e_2(\alpha)}(f_1 \circ f_2)$. Let $\alpha = (1, y_1, y_2)$. Then:

$$\begin{aligned} f_1 \circ f_2(y_1) &= (U_{(1, y_1, y_2)}(f_1 \circ f_2))(0) \\ &= [U_{(1, y_1, y_2)} f_1 \circ U_{(1, y_1, y_2)} f_2](0) \\ &= \int_{\mathcal{Q}_2^g \times \mathcal{Q}_2^g} \chi(t y_2 \cdot (u+v)) \cdot f_1(u+y_1) \cdot f_2(v+y_1) \cdot d\lambda_{u,v} \end{aligned}$$

for all $y_1, y_2 \in \mathcal{Q}_2^g$. Taking combinations of these equations for various y_2 's, it follows that

$$\int_{\mathcal{Q}_2^g \times \mathcal{Q}_2^g} g(u+v) \cdot f_1(u+y_1) \cdot f_2(v+y_2) \cdot d\lambda_{u,v} = 0$$

for all locally constant functions g such that $g(0)=0$. This shows that

$$\int h(u, v) d\lambda_{u,v} = 0$$

whenever $h \in \mathcal{H}_{2g}$, and $h(u, -u)=0$, all u . This implies that $\lambda_{u,v}$ is given by a measure on the set $\{(u, -u)\}$, i.e., there is a $\tilde{\lambda} \in \mathcal{H}_g^*$ such that

$$\int h(u, v) d\lambda_{u,v} = \int h(u, -u) d\tilde{\lambda}_u$$

all $h \in \mathcal{H}_{2g}$. Therefore:

$$f_1 \circ f_2(y) = \int_{\mathcal{Q}_2^g} f_1(y+u) f_2(y-u) d\tilde{\lambda}_u.$$

Now use the fact that $s_1 \otimes s_2(0) = s_1(0) \cdot s_2(0)$: therefore if μ and ν are the null-value measures for \mathcal{T} and $\mathcal{T}^{(2)}$, we find

$$\begin{aligned} \int_{\mathcal{Q}_2^g} f_1(u) d\mu_u \cdot \int_{\mathcal{Q}_2^g} f_2(v) d\mu_v &= \int_{\mathcal{Q}_2^g} f_1 \circ f_2(w) d\nu_w \\ &= \int_{\mathcal{Q}_2^g \times \mathcal{Q}_2^g} f_1(w+t) f_2(w-t) d\nu_w \cdot d\tilde{\lambda}_t. \end{aligned}$$

Hence if $\xi(x, y) = (x+y, x-y)$ as usual, we find

$$\begin{aligned} \int_{\mathcal{Q}_2^{2g}} F \cdot d(\mu \times \mu) &= \int_{\mathcal{Q}_2^{2g}} \xi^* F \cdot d(\nu \times \tilde{\lambda}) \quad \text{for all } F \in \mathcal{H}_{2g}, \\ \text{i. e.,} \quad \mu \times \mu(U) &= \nu \times \lambda(\xi^{-1} U) \end{aligned}$$

for all compact open sets $U \subset \mathcal{Q}_2^{2g}$. Using the evenness of μ , it follows from this equation that $\nu \times \tilde{\lambda} = \tilde{\lambda} \times \nu$, hence ν and $\tilde{\lambda}$ are proportional. Changing ν by a constant, which is permissible, we may assume $\nu = \tilde{\lambda}$. Then if ν' is the measure

$$\nu'(U) = \nu(\tfrac{1}{2} U),$$

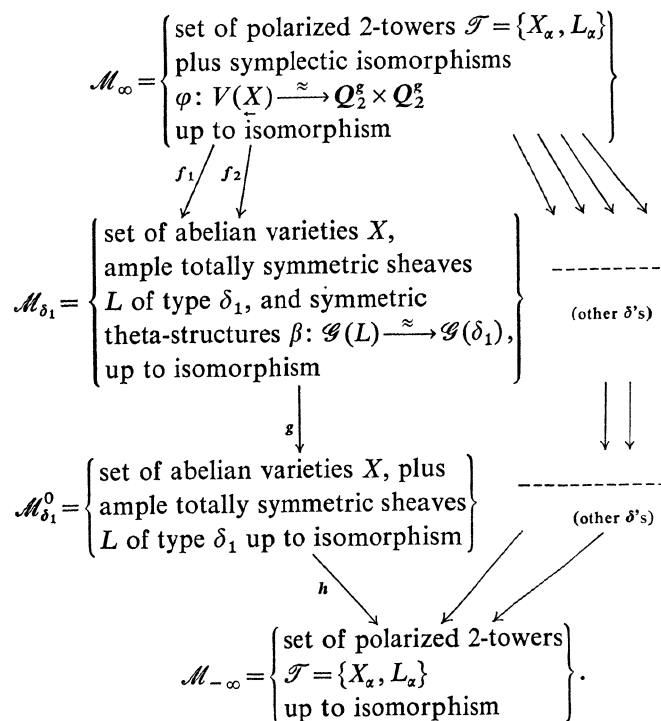
it follows that $\mu \times \mu(U) = \nu' \times \nu'(\xi U)$, all U . This is condition (A) of Lemma 2.

§ 9. The 2-Adic Moduli Space

We will now put the results of § 8 in a moduli-theoretic form, and relate these to the finite-level results of § 6. Once we have done this, we will be able to go further and determine the structure of the *boundary* of the moduli space.

The whole moduli problem for abelian varieties looks very different when viewed from an isogeny invariant point of view. The difference between polarization types disappears because any abelian variety is isogenous to a *principally polarized* abelian variety (as is easily proven by generalizing some of the results of §1 to inseparably polarized abelian varieties). The natural thing is then to view the classification in 2 steps: first one has the totality of *polarized towers*; second, within each tower one has a huge system of variously polarized abelian varieties. We will not treat this entire moduli problem since the study of all inseparably polarized abelian varieties within an isogeny type is a subject in itself. In order to a) restrict to separably polarized abelian varieties, while

b) constructing moduli schemes simultaneously in as many characteristics as possible, we shall consider only the polarized 2-tower inside each full polarized tower, and at the same time exclude only char. 2. Analogous results would be obtained if we restricted ourselves to all characteristics p , $p \nmid d$ (d a fixed *even* integer), and to isogenies within a tower of degree dividing d^N , $N \gg 0$. As far as the category of *sets* is concerned, we have the following sets and canonical maps to consider:



Here δ_1 is any g -tuple $(2^{n_1}, 2^{n_2}, \dots, 2^{n_g})$, $n_1 \geq n_2 \geq \dots \geq n_g \geq 1$. The various arrows arise as follows:

(I) g takes (X, L, β) to (X, L) ,

(II) h takes (X, L) to the 2-tower generated by (X, L) .

(III) The f_i 's are given by choosing a compact, open isotropic subgroup $U \subset Q_2^{2g}$ such that $U^\perp/U \cong H(\delta_1)$, and a symmetric isomorphism

$$\beta_0: \frac{k^* \times \Sigma(U^\perp)}{\Sigma(U)} \xrightarrow{\sim} \mathcal{G}(\delta_1).$$

Then for all (\mathcal{T}, φ) , there is a unique level $\alpha \in S$ such that $\varphi(T(\alpha)) = U$, and $f((\mathcal{T}, \varphi))$ should be $(X_\alpha, L_\alpha, \beta)$ where β is the composite

$$\mathcal{G}(L_\alpha) \xrightarrow[\text{via } \varphi]{\approx} \frac{k^* \times \Sigma(U^\perp)}{\Sigma(U)} \xrightarrow[\beta_0]{\approx} \mathcal{G}(\delta_1).$$

It is apparent from this diagram that the various moduli sets \mathcal{M}_δ treated in § 6 are inter-related in a rather complicated way: Given δ_1, δ_2 , one can choose any of an infinite number of f_1, f_2 in the diagram:

$$\begin{array}{ccc} & \mathcal{M}_\infty & \\ f_1 \swarrow & & \searrow f_2 \\ \mathcal{M}_{\delta_1} & & \mathcal{M}_{\delta_2} \end{array}$$

and relate $\mathcal{M}_{\delta_1}, \mathcal{M}_{\delta_2}$ via the, in general, many-many correspondence so obtained. Each \mathcal{M}_δ is related to itself in this way by the well-known *Hecke ring* of correspondences. The whole set-up is much easier to visualize starting from \mathcal{M}_∞ . Note that $G = \text{Sp}(2g, \mathbb{Q}_2)$ acts on \mathcal{M}_∞ , if we let $\sigma \in G$ act as follows:

$$\sigma(\mathcal{T}, \varphi) = (\mathcal{T}, \sigma \circ \varphi).$$

Then $\mathcal{M}_{-\infty}$ is nothing but the quotient \mathcal{M}_∞/G , and the \mathcal{M}_δ and \mathcal{M}_δ^0 's are quotients $\mathcal{M}_\infty/\Gamma$ where $\Gamma \subset G$ is a suitable subgroup commensurable with $\text{Sp}(2g, \mathbb{Z}_2)$. The different maps from \mathcal{M}_∞ to \mathcal{M}_δ are simply the compositions of (a) action of some $\sigma \in \text{Sp}(2g, \mathbb{Q}_2)$ and (b) the canonical map from \mathcal{M}_∞ to $\mathcal{M}_\infty/\Gamma$. Clearly the most basic sets to get ahold of are $\mathcal{M}_{-\infty}$ and \mathcal{M}_∞ . We have seen that the \mathcal{M}_δ 's "are" varieties. Thus the \mathcal{M}_∞ is an infinite covering of a variety and $\mathcal{M}_{-\infty}$ is an infinite quotient. As far as I know there is no sensible object whose underlying point set is $\mathcal{M}_{-\infty}$. But \mathcal{M}_∞ is an inverse limit of varieties and will turn out to be a perfectly upstanding (though non-noetherian) scheme. This moduli space (and its adelic generalizations) seem to be the most important ones for the entire moduli theory of abelian varieties.

The next step is to define the scheme an open subset of which will represent \mathcal{M}_∞ . We will work over the following ground ring R :

Definition 1. $R = \mathbb{Z}[\frac{1}{2}, \zeta_1, \zeta_2, \zeta_3, \dots]$ where $\zeta_n^2 = \zeta_{n-1}$ if $n \geq 2$, $\zeta_1 = -1$. The multiplicative group generated by the ζ 's is isomorphic to $\mathbb{Q}_2/\mathbb{Z}_2$ and we define:

$$\chi: \mathbb{Q}_2/\mathbb{Z}_2 \rightarrow R^*$$

via

$$\chi(m/2^n) = (\zeta_n)^m.$$

Actually, adjoining the ζ 's is not essential, but it makes life easier and seems to be quite natural.

Next, to R adjoin independent transcendentals X_α , one for each $\alpha \in \mathcal{Q}_2^{2g}$. Then divide out by the following relations:

- 1) $X_{\alpha+\beta} = \chi\left(\frac{{}^t\beta_1 \cdot \beta_2}{2}\right) \chi\left(\frac{{}^t\beta_1 \cdot \alpha_2}{2} - \frac{{}^t\alpha_1 \cdot \beta_2}{2}\right) \cdot X_\alpha$ all $\alpha \in \mathcal{Q}_2^{2g}$, $\beta \in \mathcal{Z}_2^{2g}$.
- 2) $X_{-\alpha} = X_\alpha$.
- 3) $\prod_{i=1}^4 X_{\alpha_i} = 2^{-g} \sum_{\eta \in \frac{1}{2}\mathcal{Z}_2^{2g}/\mathcal{Z}_2^{2g}} \chi({}^t\gamma_1 \cdot \eta_2 - {}^t\gamma_2 \cdot \eta_1) \cdot \prod_{i=1}^4 X_{\alpha_i + \gamma + \eta}$
all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{Q}_2^{2g}$, where $\gamma = -\frac{1}{2} \sum_{i=1}^4 \alpha_i$.

In what follows, it will be convenient to abbreviate the characters in these formulas as follows:

Definition 2.

$$e(\alpha, \beta) = \chi({}^t\alpha_1 \cdot \beta_2 - {}^t\beta_1 \cdot \alpha_2), \quad \alpha, \beta \in \mathcal{Q}_2^{2g}.$$

$$e_*(\alpha) = \chi(2{}^t\alpha_1 \cdot \alpha_2), \quad \text{if } \alpha \in \frac{1}{2}\mathcal{Z}_2^{2g}.$$

Definition 3.

$$A = R[\dots, X_\alpha, \dots] / \left\{ \begin{array}{l} \text{ideal generated by} \\ \text{relations 1, 2, 3} \end{array} \right\}.$$

$$\overline{M}_\infty = \text{Proj}(A).$$

In order to get a preliminary idea of how big \overline{M}_∞ is, introduce the subrings:

Definition 4. A_n = subring of A generated by

$$\{X_\alpha \mid 2^n \alpha \in \mathcal{Z}_2^{2g}\}.$$

Lemma 1. A is integrally dependent on A_2 .

Proof. By induction, it suffices to check that X_α is integrally dependent on A_{n-1} , when $2^n \cdot \alpha \in \mathcal{Z}_2^{2g}$ and $n \geq 2$. Use relation (3) with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, $\alpha_4 = \gamma = -\alpha$. Then since $X_{-\alpha} = X_\alpha$, we find that $X_\alpha^4 \in A_{n-1}$. *Q.E.D.*

Corollary. There are integral affine morphisms:

$$\begin{array}{ccc} \overline{M}_\infty & & \\ \pi_2 \downarrow & \searrow \pi_n & \\ & \text{Proj}(A_n) & \\ & \swarrow & \\ & \text{Proj}(A_2) & \end{array}$$

and \overline{M}_∞ is the category-theoretic \varprojlim of the algebraic schemes $\text{Proj}(A_n)$, i.e., for all schemes S ,

$$\text{Hom}(S, \overline{M}_\infty) \cong \varprojlim \text{Hom}(S, \text{Proj}(A_n)).$$

Proof. Cf. EGA, Ch. 4.

The X_α 's will be nothing but the values of the function $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ when we connect \overline{M}_∞ to the moduli problem. It is also convenient to introduce a second set of generators of the ring A , whose values will be the values of the measure μ , in the moduli problem:

For all compact open sets $U \subset \mathcal{Q}_2^g$, let

$$Y_U = 2^{-ng} \sum_{i=1}^N \sum_{\beta \in 2^{-n} \mathbb{Z}_2^g / \mathbb{Z}_2^g} \chi \left(\frac{t_{\alpha_i} \cdot \beta}{2} \right) X_{(\alpha_i, \beta)}$$

if

$$U = \bigcup_{i=1}^N [\alpha_i + 2^n \mathbb{Z}_2^g] \quad \text{and} \quad \alpha_i \not\equiv \alpha_j \pmod{2^n \mathbb{Z}_2^g}.$$

Using Lemmas 1 and 2, § 8, the relations on the X 's go over to the following relations in the Y 's:

$$1. \quad Y_{U_1} + Y_{U_2} = Y_{U_1 \cup U_2} + Y_{U_1 \cap U_2}.$$

$$2. \quad Y_{-U} = Y_U, \quad Y_\emptyset = 0.$$

3. If we define quadratic polynomials, Z_U , for all compact open sets $U \subset \mathcal{Q}_2^g \times \mathcal{Q}_2^g$, by relations (1.) and $Z_{U_1 \times U_2} = Y_{U_1} Y_{U_2}$, and if $\xi(x, y) = (x + y, x - y)$ as usual, then:

$$Z_{\xi(U_1 \times U_2)} \cdot Z_{\xi(U_3 \times U_4)} = Z_{\xi(U_1 \times U_4)} \cdot Z_{\xi(U_3 \times U_2)}.$$

In particular, let $n \geq 1$ and let

$$l: 2^{n-1} \mathbb{Z}_2^g \rightarrow \{\pm 1\}$$

be a homomorphism. Define

$$Z_{\alpha, \beta} = \sum_{\eta \in 2^{n-1} \mathbb{Z}_2^g / 2^n \mathbb{Z}_2^g} l(\eta) \cdot Y_{\alpha + \beta + \eta + 2^n \mathbb{Z}_2^g} \cdot Y_{\alpha - \beta + \eta + 2^n \mathbb{Z}_2^g}.$$

Then the general relations imply:

$$Z_{\alpha, \beta} \cdot Z_{\gamma, \delta} = Z_{\alpha, \delta} \cdot Z_{\gamma, \beta}$$

for all $\alpha, \beta, \gamma, \delta \in \mathcal{Q}_2^g$. Conversely, these relations, for all n 's, and $l \equiv +1$, imply all the quartic relations.

Moreover, the subring A_n generated by X_α , with $\alpha \in 2^{-n} \mathbb{Z}_2^{2g}$ is just the subring generated by

$$\{Y_U \mid U = U + 2^n \mathbb{Z}_2^g, U \in 2^{-n} \mathbb{Z}_2^g\}$$

or by

$$\{Y_{\alpha + 2^n \mathbb{Z}_2^g} \mid \alpha \in 2^{-n} \mathbb{Z}_2^g\}.$$

The group $G = \text{Sp}(2g, \mathbb{Q}_2)$ acts on \bar{M}_∞ in the following way:

Definition 5. For all $\sigma \in G$, let

$$(*) \quad U_\sigma(X_\alpha) = \sum_{\beta \in \mathbb{Z}_2^{2g}/\mathbb{Z}_2^{2g} \cap \sigma^{-1}(\mathbb{Z}_2^{2g})} e_*(\beta/2) e(\beta/2, \alpha) \cdot e(\gamma/2, \alpha - \beta) \cdot X_{\sigma\alpha - \sigma\beta - \sigma\gamma},$$

where $\gamma \in \mathbb{Q}_2^{2g}$ is some fixed element satisfying

$$e_*(\beta/2) \cdot e_*(\sigma\beta/2) = e(\gamma, \beta)$$

for all $\beta \in \mathbb{Z}_2^{2g} \cap \sigma^{-1}(\mathbb{Z}_2^{2g})$.

Concerning this definition, one verifies by mechanical calculation the following:

1. If the γ in the definition is varied, it must change by an element of $\mathbb{Z}_2^{2g} + \sigma^{-1}(\mathbb{Z}_2^{2g})$, and, if so, the operator U_σ is only changed into a constant multiple of itself:

$$U'_\sigma(X_\alpha) = c \cdot U_\sigma(X_\alpha), \quad \text{all } \alpha.$$

We shall assume that for each σ , some fixed γ is chosen.

Note that $-I \in \text{Sp}(2g, \mathbb{Q}_2) = G$, and $U_{-I}(X_\alpha) = X_{-\alpha}$.

2. For all $\sigma \in G$, $\alpha \in \mathbb{Q}_2^{2g}$ and $\beta \in \mathbb{Z}_2^{2g}$,

$$U_\sigma(X_{\alpha+\beta}) = e_*(\beta/2) e(\beta/2, \alpha) \cdot U_\sigma(X_\alpha).$$

Therefore, if we let M be the free R -module spanned by the X_α 's modulo *only* relations (1.), each U_σ defines an R -module homomorphism from M to M .

3. For all $\sigma, \tau \in G$, there is a non-zero constant $c_{\sigma, \tau}$ such that

$$U_\sigma \circ U_\tau = c_{\sigma, \tau} \cdot U_{\sigma \cdot \tau},$$

i.e., $\sigma \mapsto U_\sigma$ is a "projective" representation of G in M .

4. For all $\sigma \in G$, one computes easily that there is a sign $e_\sigma = \pm 1$ such that $U_{-I} \circ U_\sigma = e_\sigma U_\sigma \circ U_{-I}$. Unfortunately, it does not appear to be easy to show *directly* that $e_\sigma = +1$ for all σ . For example, if $\sigma(\mathbb{Z}_2^{2g}) = \mathbb{Z}_2^{2g}$, $e_\sigma = e_*(\gamma)$. However, using (3.), one checks that

$$e_{\sigma\tau} = e_\sigma \cdot e_\tau,$$

and since G is well known to be its own commutator subgroup, this implies $e_\sigma = +1$, all σ . But now the submodule A_1 of A of elements of degree 1 is just M modulo the span of the elements $\{X_\alpha - X_{-\alpha}\}$, and this

proves that the U_σ 's induce homomorphisms from A_1 to A_1 . We will let U_σ denote this homomorphism too.

5. The last step is to check that all the U_σ 's induce *ring* homomorphisms from A to A . Frankly, I balked at directly applying U_σ to the relations (3.) and seeing what comes out. But, in 2 special cases, it isn't too bad. Suppose first that σ is in the subgroup $\Gamma = \text{Sp}(2g, \mathbb{Z}_2)$. Then $\sigma(\mathbb{Z}_2^g) = \mathbb{Z}_2^g$, and U_σ reduces to:

$$U_\sigma(X_\alpha) = e(\gamma/2, \alpha) \cdot X_{\sigma\alpha - \sigma\gamma}.$$

In this case, it's not hard to check that U_σ takes a relation of type (3.) to another relation of the same type, so that U_σ induces a map from A to A . Suppose second that σ is in the sub-semi-group:

$$H^+ = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & {}^t A^{-1} \end{array} \right) \mid \begin{array}{l} A \in GL(g, \mathbb{Q}_2) \\ A(\mathbb{Z}_2^g) \supseteq \mathbb{Z}_2^g \end{array} \right\} \subset G.$$

For such σ , U_σ reduces to:

$$U_\sigma(X_\alpha) = \sum_{\beta \in \mathbb{Z}_2^g / A^{-1} \mathbb{Z}_2^g} e(-\beta_1/2, \alpha) \cdot X_{\sigma\alpha + \sigma\beta_1}$$

(where β_1 is the $2g$ -vector $(\beta, 0)$). Now, referring back to the proof of Lemma 2, in the last section, we see that an equivalent form of the relations (3.) is:

$$Y_{\alpha, \beta} \cdot Y_{\gamma, \delta} = Y_{\alpha, \delta} \cdot Y_{\gamma, \beta}, \quad \text{all } \alpha, \beta, \gamma, \delta \in \mathbb{Q}_2^{2g}$$

where

$$Y_{\alpha, \beta} = \sum_{\eta \in \frac{1}{2} \mathbb{Z}_2^g / \mathbb{Z}_2^g} e(-\eta_1, \alpha) X_{\alpha + \beta + \eta_1} \cdot X_{\alpha - \beta + \eta_1}.$$

For $\sigma \in H^+$, $Y_{\alpha, \beta}$ behaves very nicely. One computes easily that:

$$U_\sigma(Y_{\alpha, \beta}) = \sum_{\eta, \zeta \in \frac{1}{2} A \mathbb{Z}_2^g / \frac{1}{2} \mathbb{Z}_2^g} e(-\eta_1, \sigma\alpha) \cdot e(-\zeta_1, \sigma\beta) \cdot Y_{\sigma\alpha + \eta_1, \sigma\beta + \zeta_1}.$$

From this it follows immediately that $U_\sigma[Y_{\alpha, \beta} \cdot Y_{\gamma, \delta} - Y_{\alpha, \delta} \cdot Y_{\gamma, \beta}]$ is an R -linear combination of expressions $Y_{\alpha', \beta'} \cdot Y_{\gamma', \delta'} - Y_{\alpha', \delta'} \cdot Y_{\gamma', \beta'}$.

Finally, it follows from the paper of IWAHORI-MATSUMOTO [13] that all the double cosets in $\Gamma \backslash G / \Gamma$ are represented by matrices:

$$\sigma = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & {}^t A^{-1} \end{array} \right)$$

$$A = \begin{pmatrix} 2^{a_1} & & & 0 \\ & 2^{a_2} & & \\ & & \ddots & \\ 0 & & & 2^{a_g} \end{pmatrix}$$

$$0 \geq a_1 \geq a_2 \geq \dots \geq a_g.$$

Since there are in H^+ , $G = \Gamma \cdot H^+ \cdot \Gamma$ and our 2 calculations suffice to prove that for every $\sigma \in G$, U_σ maps A to A .

Putting all this together, we conclude that Definition 5 defines a projective representation of G on A , and an action of G on the scheme \bar{M}_∞ . In fact, let $V_\sigma: \bar{M}_\infty \rightarrow \bar{M}_\infty$ denote this action, then, by definition,

$$V_\sigma^*(X_\alpha) = U_{\sigma^{-1}}(X_\alpha)$$

(the σ^{-1} makes it an action of G instead of the opposed group).

Now we can connect \bar{M}_∞ to the moduli problem. For all algebraically closed fields k ($\text{char}(k) \neq 2$), there is a map:

$$\Theta: \mathcal{M}_\infty(k) \rightarrow \left\{ \begin{array}{l} \text{Set of } k\text{-valued} \\ \text{points of } \bar{M}_\infty \end{array} \right\}$$

(\mathcal{M}_∞ denotes the set defined at the beginning of this §), which assigns to a tower \mathcal{T} and a $\varphi: V(\underline{X}) \xrightarrow{\sim} \mathcal{Q}_2^{2g}$, the point with homogeneous coordinates

$$X_\alpha = \mathfrak{g} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\varphi^{-1} \alpha), \quad \text{or} \quad Y_U = \mu(U)$$

(cf. Lemma 1 and 2, § 8).

Recall that G acts on $\mathcal{M}_\infty(k)$ as follows:

Definition 6. Let $(\mathcal{T}, \varphi) \in \mathcal{M}_\infty(k)$ and $\sigma \in G$. Then let $U_\sigma((\mathcal{T}, \varphi))$ be the pair $(\mathcal{T}, \sigma \circ \varphi)$, i.e., modify the symplectic isomorphism σ to:

$$V(\underline{X}) \xrightarrow[\varphi]{\sim} \mathcal{Q}_2^{2g} \xrightarrow[\sigma]{\sim} \mathcal{Q}_2^{2g}.$$

Theorem 1. Under Θ , the actions of G on $\mathcal{M}_\infty(k)$ and on the set of k -valued points of \bar{M}_∞ are compatible.

Proof. Recall that $\mathfrak{g} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the unique element f of $\mathfrak{g}(\Gamma(\mathcal{T}))$ invariant satisfying

$$\text{a)} \quad f(x+a) = e_*(a/2) \cdot e_\lambda(a/2, x) \cdot f(x)$$

for all $a \in \varphi^{-1}(\mathcal{Z}_2^{2g})$, with $e_*(\varphi^{-1}(\alpha)) = \chi(2^t \alpha_1 \cdot \alpha_2)$, all $\alpha \in \frac{1}{2} \mathcal{Z}_2^{2g}$. If we use the symplectic isomorphism $\sigma \circ \varphi$, the new function $\mathfrak{g} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ will instead be the unique f in $\mathfrak{g}(\Gamma(\mathcal{T}))$ satisfying:

$$\text{a')} \quad f(x+a) = e'_*(a/2) \cdot e_\lambda(a/2, x) \cdot f(x)$$

for all $a \in \varphi^{-1}(\sigma^{-1}(\mathcal{Z}_2^{2g}))$, with $e'_*(\varphi^{-1}(\sigma^{-1}(\alpha))) = \chi(2^t \alpha_1 \cdot \alpha_2)$ all $\alpha \in \frac{1}{2} \mathcal{Z}_2^{2g}$.

Let $\mathfrak{g}^\sigma \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ denote this new $\mathfrak{g} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then

$$X_\alpha[U_\sigma(\mathcal{T}, \varphi)] = \mathfrak{g}^\sigma \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\varphi^{-1}(\alpha)).$$

To find $\mathfrak{g}^\sigma \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, recall that the functions

$$\alpha \mapsto e_\lambda(\beta/2, \alpha) \cdot \mathfrak{g} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\alpha - \beta), \quad \beta \in V(\underline{X}).$$

span $V(\underline{X})$. Therefore, it suffices to find some linear combination of these functions satisfying a'). If we make all these into functions on \mathcal{Q}_2^{2g} via $\sigma \circ \varphi: V(\underline{X}) \xrightarrow{\sim} \mathcal{Q}_2^{2g}$, we find that

$$\mathfrak{g}^\sigma \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\varphi^{-1}(\sigma^{-1}(\alpha)))$$

is that linear combination f of the functions

$$g_\beta(\alpha) = e(\beta/2, \alpha) \cdot g(\sigma^{-1}\alpha - \sigma^{-1}\beta), \quad \beta \in \mathcal{Q}_2^{2g}$$

$$g(\alpha) = \mathfrak{g} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\varphi^{-1}\alpha)$$

such that

$$\text{a'')} \quad f(\alpha + \beta) = e_*(\beta/2) \cdot e(\beta/2, \alpha) \cdot f(\alpha),$$

for all $\beta \in \mathcal{Z}_2^{2g}$. One solution of (a'') is the function:

$$\begin{aligned} g^\sigma(\alpha) = & \sum_{\beta \in \mathcal{Z}_2^{2g}/\mathcal{Z}_2^{2g} \cap \sigma \mathcal{Z}_2^{2g}} e_*(\beta/2) \cdot e(\beta/2, \alpha) \cdot e(\gamma/2, \alpha - \beta) \\ & \times g(\sigma^{-1}\alpha - \sigma^{-1}\beta - \sigma^{-1}\gamma) \end{aligned}$$

where γ satisfies $e_*(\beta/2) \cdot e_*(\sigma^{-1}\beta/2) = e(\gamma, \beta)$, all $\beta \in \mathcal{Z}_2^{2g} \cap \sigma \mathcal{Z}_2^{2g}$. If this function is *not* 0, it must equal

$$\mathfrak{g}^\sigma \begin{bmatrix} 0 \\ 0 \end{bmatrix} \circ \varphi^{-1} \circ \sigma^{-1}.$$

But on the other hand, X_α has values $g(\alpha)$ at the geometric point x corresponding to (\mathcal{T}, φ) . Hence X_α has the values at $V_\sigma(x)$:

$$\begin{aligned} X_\alpha(V_\sigma(x)) &= V_\sigma^* X_\alpha(x) \\ &= U_{\sigma^{-1}} X_\alpha(x) \\ &= \sum_{\beta \in \mathcal{Z}_2^{2g}/\mathcal{Z}_2^{2g} \cap \sigma \mathcal{Z}_2^{2g}} e_*(\beta/2) \cdot e(\beta/2, \alpha) \times \\ & \quad \times e(\gamma/2, \alpha - \beta) X_{\sigma^{-1}\alpha - \sigma^{-1}\beta - \sigma^{-1}\gamma}(x) \\ &= g^\sigma(\alpha). \end{aligned}$$

Therefore, $g^\sigma \neq 0$, so $g^\sigma(\alpha)$ is also the value of X_α at the geometric point corresponding to $(\mathcal{T}, \sigma \circ \varphi)$. *Q.E.D.*

Now every moduli space is supposed to represent a functor. In our case, instead of making a big fuss over defining a *family* of 2-towers of abelian varieties, over a scheme S , it is simpler to observe that $\mathcal{M}_\infty(k)$ is an inverse limit of some of the $\mathcal{M}_\delta(k)$'s introduced in § 6, and then to define $\mathcal{M}_\infty(S)$ as the corresponding limit of these $\mathcal{M}_\delta(S)$'s.

Given a polarized 2-tower $\mathcal{T} = (\{X_\alpha\}, \{L_\alpha\})$ and $\varphi: V(X) \xrightarrow{\sim} \mathcal{Q}_2^{2g}$, for all $n \geq 1$, we get

a) an abelian variety $X_n = X_{\alpha_n}$, where

$$T(\alpha_n) = \varphi^{-1}(2^{-n} \mathbf{Z}_2^{2g}),$$

b) a totally symmetric ample invertible sheaf

$$L_n = L_{\alpha_n} \text{ on } X_n, \text{ of type } \delta_n = (2^{2n}, 2^{2n}, \dots, 2^{2n}),$$

c) a symmetric \mathfrak{g} -structure:

$$\lambda_n: \mathcal{G}(L_n) \cong \frac{\mathcal{G}_{\alpha_n}^*(\mathcal{T})}{K(\alpha_n)} \underset{\text{via } \varphi}{\cong} \frac{k^* \times 2^{-n} \mathbf{Z}_2^{2g}}{\{1\} \times 2^n \mathbf{Z}_2^{2g}} \cong \mathcal{G}(\delta_n).$$

This is a map

$$(\mathcal{T}, \varphi) \mapsto (X_n, L_n, \lambda_n)$$

$$\mathcal{M}_\infty(k) \rightarrow \mathcal{M}_{\delta_n}(k).$$

Note that all these X_n 's are canonically isomorphic, e.g., to X_1 , via diagrams:

$$\begin{array}{ccc} X_n & & \\ \parallel & \searrow^{p_{\alpha_n, \alpha_1}} & \\ X_1 & \xrightarrow{2^{n-1} \delta} & X_1 \end{array}$$

and that under these isomorphisms, L_n is just $L_1^{4^{n-1}}$.

On the other hand, between the functors $\mathcal{M}_{\delta_{n+1}}$ and \mathcal{M}_{δ_n} , we have a natural transformation:

$$\pi_n: \mathcal{M}_{\delta_{n+1}} \rightarrow \mathcal{M}_{\delta_n}.$$

In fact, given \mathcal{X}/S , L , and λ in $\mathcal{M}_{\delta_{n+1}}(S)$, we get

a) a second ample sheaf M on \mathcal{X} , by descending L with respect to the isogeny

$$2\delta: \mathcal{X} \rightarrow \mathcal{X}$$

and the descent data $\lambda^{-1}(K_2)$, where $K_2 \subset \mathcal{G}(\delta_{n+1}) \times S$ is the subscheme representing the subfunctor of triples $(1, x, l)$, $2x = 2l = 0$.

b) Since

$$\begin{aligned} \mathcal{G}(M) &\cong \frac{\text{normalizer of } \lambda^{-1}(K_2) \text{ in } \mathcal{G}(L)}{\lambda^{-1}(K_2)} \\ &\cong \frac{\text{normalizer of } K_2 \text{ in } \mathcal{G}(\delta_{n+1})}{K_2} \cong \mathcal{G}(\delta_n) \end{aligned}$$

we get a \mathfrak{g} -structure $\mu: \mathcal{G}(M) \xrightarrow{\sim} \mathcal{G}(\delta_n)$.

It can be checked that $(\mathcal{X}/S, M, \mu) \in \mathcal{G}_{\delta_n}(S)$, so we call this $\pi_n((\mathcal{X}/S, L, \lambda))$.

Going back to k -valued points, we have a diagram:

$$\begin{array}{ccc} & \vdots & \\ & \downarrow & \\ \mathcal{M}_{\infty}(k) & \xrightarrow{\quad} & \mathcal{M}_{\delta_{n+1}}(k) \\ & \searrow & \downarrow \pi_n \\ & & \mathcal{M}_{\delta_n}(k) \\ & & \vdots \\ & & \downarrow \\ & & \mathcal{M}_{\delta_1}(k) \end{array}$$

and it is clear that this induces an *isomorphism*

$$\mathcal{M}_{\infty}(k) \cong \varprojlim \mathcal{M}_{\delta_n}(k).$$

Definition 7. For all schemes S , let $\mathcal{M}_{\infty}(S) = \varprojlim \mathcal{M}_{\delta_n}(S)$.

Next, let's translate the results of § 6 on the representability of \mathcal{M}_{δ_n} into the discussion. The results there show that there is an open set:

$$M_{\delta_n} \subset \text{Proj}(A_n^*) = \overline{M}_{\delta_n}$$

$$A_n^* = R[\dots, Q^{(n)}(a), \dots] / \left\{ \begin{array}{l} \text{Modulo } Q^{(n)}(a) = Q^{(n)}(-a) \\ \text{and certain quartic relations} \end{array} \right\} \quad a \in K(\delta_n)$$

such that M_{δ_n} represented \mathcal{M}_{δ_n} (over the ring R). Moreover, in Step II of the proof, a canonical morphism

$$\pi: \overline{M}_{2\delta} \rightarrow \overline{M}_{\delta}$$

was introduced. Iterated, this defines a morphism from $\overline{M}_{4\delta} \rightarrow \overline{M}_\delta$, e. g., from $\overline{M}_{\delta_{n+1}}$ to \overline{M}_{δ_n} . If you work it out, it is just the projection:

$$\begin{array}{ccc} \overline{M}_{\delta_{n+1}} & \xrightarrow{\pi_n} & \overline{M}_{\delta_n} \\ \parallel & & \parallel \\ \text{Proj}(A_{n+1}^*) & & \text{Proj}(A_n^*), \end{array}$$

$$\pi_n^*(Q^{(n)}(a)) = \sum_{\substack{2b=a \\ b \in K(\delta_{n+1})}} Q^{(n+1)}(b), \quad \text{all } a \in K(\delta_n)$$

(here we identify $K(\delta_n)$ with a subgroup of $K(\delta_{n+1})$ as before). In particular, π_n is a finite morphism. Moreover, as we saw in § 6, (Step II).

(a) π_n is étale at all points of $M_{\delta_{n+1}}$ and $\pi_n^{-1}(M_{\delta_n}) = M_{\delta_{n+1}}$. In fact:

(b) The morphism of schemes $\pi_n: M_{\delta_{n+1}} \rightarrow M_{\delta_n}$ corresponds to the transformation of functors $\pi_n: \mathcal{M}_{\delta_{n+1}} \rightarrow \mathcal{M}_{\delta_n}$ via the representability proven in § 6.

(This is easy to check and we omit the proof.)

Now, passing to the limit over the homomorphisms π_n^* , note that the direct limit of the rings A_n^* is just A itself. In fact,

$$K(\delta_n) = \bigoplus_{g \text{ times}} \mathbf{Z}/2^{2^n} \mathbf{Z},$$

and let

$$Q^{(n)}(a_1, \dots, a_g) \mapsto Y_{(2^{-n}a_1, \dots, 2^{-n}a_n) + 2^n \mathbf{Z}_2^g}$$

($a_i \in \mathbf{Z}/2^{2^n} \mathbf{Z}$). It is easy to check that the relations imposed on the $Q^{(n)}(a)$'s give us exactly the defining relations on the Y_U 's. Moreover, A_n is just the image of A_n^* in A . (It may very well happen that each π_n^* is injective, so that $A \cong \bigcup_n A_n^*$ and $A_n \cong A_n^*$: but I have no proof for this.) Geometrically, this shows that

$$\overline{M}_\infty \cong \varprojlim_n \overline{M}_{\delta_n}.$$

Definition 8. Let M_∞ be the inverse image in \overline{M}_∞ of M_{δ_n} in \overline{M}_{δ_n} (independent of n by (a)).

1. $M_\infty \cong \varprojlim_n M_{\delta_n}$
2. $\mathcal{M}_\infty(S) \cong \varprojlim_n \mathcal{M}_{\delta_n}(S)$
3. The M_{δ_n} 's represent \mathcal{M}_{δ_n} (compatibly as n varies). Hence:

Theorem 2. The scheme M_∞ represents \mathcal{M}_∞ .

Recapitulating the discussion, the basic set which we are classifying is:

$$\begin{aligned} \mathcal{M}_\infty(k) &= \left\{ \begin{array}{l} \text{set of polarized 2-towers } \mathcal{T} = (X_\alpha, L_\alpha) \\ \text{plus symplectic isomorphisms } \varphi: V(\mathcal{T}) \xrightarrow{\sim} \mathbf{Q}_2^{2g} \end{array} \right\} \\ &= \left\{ \begin{array}{l} \text{set of abelian varieties } X, \text{ totally} \\ \text{symmetric ample } L \text{ on } X \text{ of type } (4, 4, \dots, 4), \\ \text{and symmetric } \mathfrak{g}\text{-structures } \lambda_n \text{ on } \mathcal{G}((2^n \delta)^* L), \\ \text{for all } n, \text{ which are "compatible" as } n \text{ varies} \end{array} \right\}. \end{aligned}$$

Quite clearly, we can also say:

$$\mathcal{M}_\infty(k) = \left\{ \begin{array}{l} \text{set of abelian varieties } X, \text{ totally symmetric} \\ \text{ample } L \text{ on } X \text{ of type } (4, 4, \dots, 4), \text{ and symplectic} \\ \text{isomorphisms } \varphi: V(X) \xrightarrow{\sim} \mathbf{Q}_2^{2g} \text{ such that } \varphi(T(X)) = 2\mathbf{Z}_2^{2g} \end{array} \right\},$$

or:

$$\mathcal{M}_\infty(k) = \left\{ \begin{array}{l} \text{set of abelian varieties } X, \text{ symmetric ample } L \text{ on } X \text{ of} \\ \text{degree 1, and symplectic isomorphisms: } \varphi: V(X) \xrightarrow{\sim} \mathbf{Q}_2^{2g} \\ \text{such that } \varphi(T(X)) = \mathbf{Z}_2^{2g}, \text{ and } e_*^L(\varphi^{-1}(\alpha)) = \chi(2^t \alpha_1 \cdot \alpha_2) \text{ all} \\ \alpha \in \frac{1}{2} \mathbf{Z}_2^{2g} \end{array} \right\}.$$

In any case, the principal result, on the level of geometric points, is:

Corollary. *For all algebraically closed fields k , the map Θ is a bijection between $\mathcal{M}_\infty(k)$ and the set of k -valued points of the open set M_∞ .*

In other words, the whole tower \mathcal{T} , plus $\varphi: V(X) \xrightarrow{\sim} \mathbf{Q}_2^{2g}$, is determined by the theta function

$$\mathfrak{g} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \circ \varphi^{-1}$$

(or the measure μ) and the theta functions that arise in this way are those which satisfy some finite set of inequalities. Our next task is to determine these inequalities, and hence M_∞ , explicitly.

Department of Mathematics
Harvard University
Cambridge, Massachusetts

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